The Khovanov Homology of Rational Tangles

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Abstract

We show that the Khovanov complex of a rational tangle has a very simple representative whose backbone of non-zero morphisms forms a zig-zag. We find that the bigradings of the subobjects of such a representative can be described by matrix actions, which after a change of basis is the reduced Burau representation of B_3.

1 Introduction

At the turn of the century, Khovanov constructed a homology theory of links in $\mathbb{R}^3$ which categorified the Jones polynomial [Kho99]. Notable for its method of construction and functorality amoungst other properties, Khovanov homology has been shown to detect the unknot [KM10], and has been used to provide a combinatorial proof of the Milnor conjecture [Ras04]. Though originally defined for links, the theory has been extended to tangles [Kho01], and has seen alternative formulations [Bar04].

Rational tangles are a simple kind of tangle, but are not simple enough as to be boring. Defined by Conway in [Con70], these are essentially formed by applying a series of twists to the ends of a tangle, beginning with two straight strands (see Figure 2.1.1).

While the structure of the Khovanov complex $[\![T]\!]$ of a tangle T in general admits no easy description, in the case that T is rational, we show that $[\![T]\!]$ has a representative that is a zig-zag complex.

We rigourously define what we mean by a zig-zag complex below, but the core idea is that the backbone of non-zero maps between the subobjects of the complex collectively form a zig-zag. (See for instance, Figure 4.3.1.)

Consider, for example, the following complex.

$$\bullet \to \mathbb{Z} \xrightarrow{\begin{pmatrix} 0 \\ 2 \end{pmatrix}} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\begin{pmatrix} 4 & 0 \\ 3 & 2 \end{pmatrix}} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\begin{pmatrix} 2 & 0 \end{pmatrix}} \mathbb{Z} \to \bullet$$

When we explicitly draw out the non-zero morphisms in the matrices, these form a zig-zag.

$$\begin{tikzcd}
& \mathbb{Z} \arrow[r, "4"] \arrow[ddr, "3"] & \mathbb{Z} \arrow[dr, "2"] & \\
\mathbb{Z} \arrow[dr, "2"] & \oplus & \oplus & \mathbb{Z} \\
& \mathbb{Z} \arrow[r, "2"] & \mathbb{Z} &
\end{tikzcd}$$

Definition 1.0.1 Let $\mathcal{C}$ be a preadditive category, and let $\mathrm{Mat}(\mathcal{C})$ be the additive closure of $\mathcal{C}$ (Definition 3.1.2). Let (Ω, ∂) be a representative of a complex in $\mathrm{Kom}(\mathrm{Mat}(\mathcal{C}))$, the

category of complexes in $\mathrm{Mat}(\mathcal{C})$ considered up homotopy equivalence (Definition 3.1.3). We will define a *zig-zag* complex using the following definitions:

Let ∂^i_j denote the differential ∂^i restricted to the subobject Ω^i_j in $\oplus_k \Omega^i_k = \Omega^i$.

Call a subobject Ω^i_j a *z-end* of the complex if either:

- every map from the subobjects of Ω^{i-1} to Ω^i_j is zero, and there is precisely one non-zero map from Ω^i_j to a subobject of Ω^{i+1}; or
- there is precisely one non-zero map from a subobject of Ω^{i-1} to Ω^i_j, and $\partial^i_j = 0$.

Call a subobject Ω^i_j a *z-middle* of the complex if either:

- there is precisely one non-zero map from a subobject of Ω^{i-1} to Ω^i_j and there is precisely one non-zero map from Ω^i_j to a subobject of Ω^{i+1}; or
- every map from the subobjects of Ω^{i-1} to Ω^i_j is zero and there are precisely two non-zero maps from Ω^i_j to subobjects of Ω^{i+1}; or
- there are precisely two non-zero maps from subobjects of Ω^{i-1} to Ω^i_j and $\partial^i_j = 0$.

Then we say (Ω, ∂) is a *zig-zag complex* if every subobject in the complex is either a z-end or a z-middle, and precisely two subobjects are z-ends. A complex in $\mathrm{Kom}(\mathrm{Mat}(\mathcal{C}))$ is a zig-zag complex if it has a representative which is a zig-zag complex.

The structure of a Khovanov complex is only half the picture though: Khovanov homology is a bigraded theory – the position of a subobject in the complex equips it with a homological grading and the subobject also picks up an internal (quantum) grading. While the bigrading information of $[\![T]\!]$ generally defies a concise description, it turns out that the bigrading information of $[\![T]\!]$ for rational T does: when crossings are added to the underlying tangle, the changes in the bigrading information of the complex can be described with matrix actions. These actions are, after a change of basis, the reduced Burau representation of B_3. While we expected to find an action describing the quantum gradings, that we could describe both the quantum and homological gradings was a surprise. The ramifications of this observation remain unclear.

The structure of the paper is as follows. In Section 2 we review rational tangles, following [KL03b]. Readers familiar with these topological objects should look at the diagrams and examples to acquaint themselves with our notation, but should otherwise feel free to skip the section. For everyone else, we discuss their basic structure, and explain how rational tangles are classified by a function known as the tangle fraction. This function associates to each rational tangle a rational number.

In Section 3 we review the Khovanov homology theory we will apply to rational tangles. Experts familiar with Bar-Natan's dotted cobordism theory (in particular with [Bar06]) should feel free to skip the section. We describe Bar-Natan's generalization of Khovanov homology to tangles [Bar04], then specialize this to his 'dotted' theory we apply in the next section. On links, the dotted theory is equivalent to Khovanov's original theory in [Kho99] but is significantly easier to work with. We demonstrate this by providing a quick proof of the invariance of the Khovanov bracket under the Reidemeister moves.

In Section 4, we combine the previous topics to develop a theory of the Khovanov homology of rational tangles. We determine the Khovanov complexes of integer tangles, before presenting a helpful isomorphism and discussing its implications. We then combine these to prove Theorem 4.3.1, one of our main results. This theorem explicitly describes how the Khovanov complex changes when crossings are added to the underlying tangle, from which it follows that the Khovanov complex of a rational tangle is a zig-zag complex.

In Section 5 we use the proof of Theorem 4.3.1 to show that the bigrading information of the Khovanov complexes of rational tangles can be described by matrix actions. This is Theorem 5.0.4, our other main result. After a change of basis, this action is found to be the reduced Burau representation of B_3, the three strand braid group.

Acknowledgements

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All but one of the figures in this paper were created with TikZ (Figure 4.3.1 was created with `gnuplot`). As such kudos must go to Till Tantau and everyone else who's worked on the package. Anyone who's used it knows the pain involved; as such a separate file containing the code for all the figures can be found in the arXiv folder for this paper, should anyone find them useful. I make no apologies for the state of the code therein.

2 Rational tangles

In this section we review the main parts to the theory of rational tangles. In particular, we explain how rational tangles are classified by the tangle fraction, a function that takes each rational tangle to a rational number.

In Section 2.1 we define rational tangles (Definition 2.1.5), and discuss some of their basic properties. The material in this subsection, like the rest of the section, is adapted from [KL03b].

In Section 2.2 we define the tangle fraction (Definition 2.2.4). This function classifies rational tangles, and is the reason why rational tangles are called rational: two rational tangles are isotopic iff their tangle fraction is the same rational number.

In Section 2.3 we briefly examine rational links. These are links obtained by closing the ends of rational tangles, and many of the simplest examples of knots can be obtained in this manner. (Of the first 25 knots, 24 are rational.) The classification of rational tangles with the tangle fraction can be used to classify rational links. This theorem provides many elegant descriptions of properties such as chirality and invertibility for rational links in terms of their tangle fractions.

2.1 Rational tangles

Although rational tangles can be defined quite generally (Definition 2.1.1), they satisfy a surprising set of isotopies (Proposition 2.1.9), which allows us to write each in a *standard form* (Definition 2.1.12). In this subsection we define the the standard form of a rational tangle, a description that is much easier to understand than more abstract definitions of rational tangles.

The notation we develop in this section will used extensively in Section 4 when we describe the Khovanov homology of rational tangles.

Before we jump into the theory of rational tangles, we first describe tangles in general.

Definition 2.1.1 A *tangle* is an smooth embedding of a compact 1-manifold X, possibly with boundary, into the three-ball B^3 such that the boundary of X is sent to the boundary of B^3 transversely. By a smooth embedding we mean an injective smooth map whose differential is nowhere zero.

We will identify tangles with their image in B^3. Tangles may also be oriented – an oriented tangle is a tangle where every connected component has the choice of one of the two possible directions on it. We will consider tangles up to isotopy.

Definition 2.1.2 Two tangles T_1, T_2 are said to be *isotopic*, denoted $T_1 \sim T_2$, if one can be transformed into the other via a smooth isotopy fixing the endpoints. That is, via a smooth map $F : X \times [0, 1] \to B^3$ such that:

- $F(X, t)$ is a tangle for each t,
- $F(X, 0) = T_1$, $F(X, 1) = T_2$,
- $F(\partial X, t)$ is constant with respect to t.

The corresponding notion of isotopy for oriented tangles is clear.

We will depict tangles via *tangle diagrams*. That is, we take a regular projection (see below) of the tangle onto a plane whose intersection with B^3 is a great circle, and at each crossing point in the projection (a point whose preimage consists of two points) mark the overcrossings and undercrossings of the tangle in the obvious way. That is, after projection, we mark which branches of the tangle are "higher" and "lower" than the others. Examples of tangle diagrams can be found below in Figure 2.1.2.

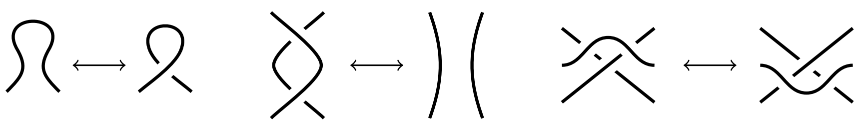
By a regular projection we mean:

1. the boundary points of the tangle are sent to distinct points on the great circle and non-boundary points are sent inside the circle;
2. lines tangent to the tangle, whenever defined, are projected onto lines in the plane;
3. no more than two points of the link are projected to the same point of the plane;
4. the set of crossing points is finite, and at each crossing point the projections of the tangents to the points in the preimage do not coincide.

Oriented tangle diagrams are defined in the same way, only now one marks the orientations of the connected components with arrows.

Isotopic tangles and their diagrams are related by a well-known Theorem due to Reidemeister [Rei48].

Theorem 2.1.3 *Two tangles S, T are isotopic iff they have the same endpoints and any tangle diagram of S can be transformed into a tangle diagram of T via planar isotopies fixing the boundary of the projection disk and a finite sequence of local moves taking place in the interior of the disk of the following three types:*



R1 *R2* *R3*

This allows us to think about tangles almost entirely in terms of tangle diagrams. We now introduce the central object of this section.

Definition 2.1.4 A *rational tangle* is a smooth embedding of two copies of the unit interval I_1, I_2 into B^3 (in which the boundary of $I_1 \sqcup I_2$ is sent to the boundary of B^3 transversely), such that there exists a homeomorphism of pairs

$$h : (B^3, I_1 \sqcup I_2) \to (D^2 \times I, \{x, y\} \times I).$$

This definition is fairly abstract, and not as easy to understand as our next definition of a rational tangle (Definition 2.1.5), but we have nonetheless included it for completeness. The definition is equivalent to saying that rational tangles are isotopic to tangles obtained by applying a finite number of consecutive twists to neighbouring endpoints of two untangled arcs, illustrated in Figure 2.1.1 below.

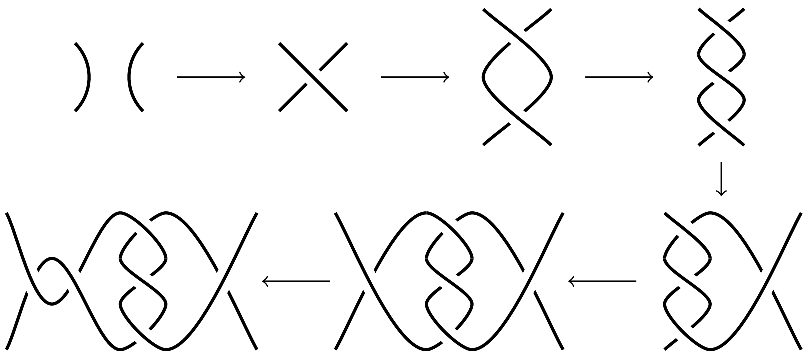


Figure 2.1.1: Constructing a rational tangle. Beginning with two untangled arcs, various twists of neighbouring endpoints are applied. Each arrow represents a twist; the first three arrows are twists of the top or bottom endpoints, the others are twists of the left and right endpoints.

It will be convenient to have a concise notation to describe the structure of rational tangles, which we now develop.

The simplest rational tangles are those which admit a crossingless diagram; we call these the $[0]$ and $[\infty]$ tangles. The next simplest are the *integer tangles* $[n]$ and the *vertical tangles* $[\overline{n}]$, made of n horizontal or vertical twists respectively. Their tangle diagrams are in Figure 2.1.2 below.

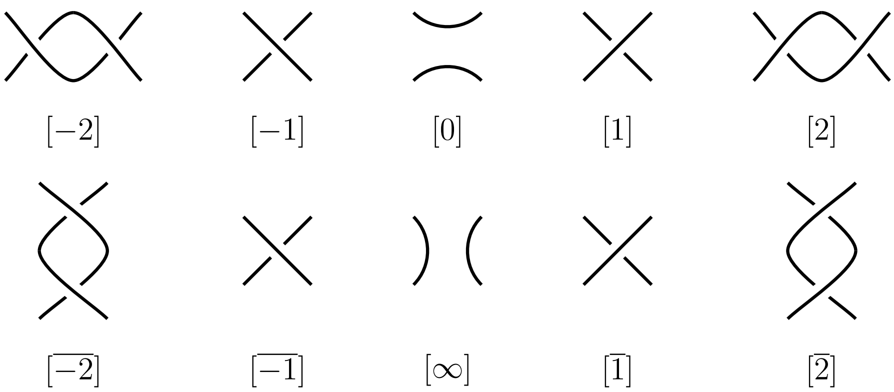


Figure 2.1.2: Some simple integer $[n]$ and vertical $[\overline{n}]$ tangles.

Note the sign conventions: if n is positive, so are the gradients of the overcrossings in the tangle diagrams above. These are the same conventions that Conway used in [Con70], but different to those used by Kauffman and Lambropoulou in [KL03b].

Rational tangles are a subclass of tangles with four boundary points, called *4 boundary point tangles* or just *4 point tangles*. On 4 point tangles one can define binary operations referred to as *addition* (+) and *multiplication* ($*$). The definition of $T + S$ and $T * S$ for arbitrary 4 point tangles is given in Figure 2.1.3 below. The operations are well-defined up to isotopy.

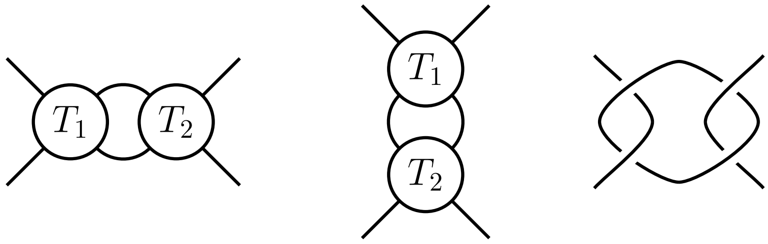


Figure 2.1.3: LEFT The sum (+) of 4 point tangles T and S. MIDDLE The product ($*$) of 4 point tangles T and S. RIGHT The tangle $[\bar{2}] + [\bar{2}]$ is not rational, since it contains three connected components.

Note that addition and multiplication of 4 point tangles are associative but not commutative. The sum or product of rational tangles is not necessarily rational.

Using these operations and the $[\pm 1]$ tangles illustrated above, we can give an alternative, and much more usable, definition of a rational tangle.

Definition 2.1.5 A tangle is *rational* if it can be created from $[0]$ or $[\infty]$ by a finite sequence of additions and multiplications with the tangles $[\pm 1]$. If a rational tangle is presented in this way we say it is in *twist form*.

For instance, the rational tangle constructed in Figure 2.1.1 is in twist form, since it is given by $[-1] + [-1] + ((([\infty] * [-1] * [-1] * [-1]) + [1])$.

We omit the proof of the equivalence of this definition and that of Definition 2.1.4, but refer the interested reader to Note 1 of [KL03b]. Unless otherwise stated, in the sequel we assume that any rational tangle is in twist form.

One can also define operations on 4 point tangles T called *mirror image*, *rotation* and *inversion*. The mirror image of T is the tangle obtained from T by switching all the crossings, and is denoted $-T$. The rotation of T is obtained by rotating T by $\pi/2$ counterclockwise and is denoted T^r. Finally, the inversion of T is defined by $T^{-1} := -T^r$. Examples of the mirror image and inverse of a tangle are below.

Figure 2.1.4: LEFT A tangle and its mirror image. RIGHT A tangle and its inverse.

The reasons why the first two operations are called the mirror image and rotation of a tangle are self-explanatory, but it may be unclear why their composition should be called the inverse. This is because for rational tangles, the tangle fraction respects the the inverse of a tangle. That is, $F(T^{-1}) = F(T)^{-1}$. We describe the tangle fraction satisfies in the next section.

We now define a type of isotopy, and use it to show some surprising properties of rational tangles. These in turn allow us to simplify the definition of a rational tangle.

Definition 2.1.6 A *flype* is an isotopy of a tangle applied to a 4 point subtangle of the form $[\pm 1]+T$ or $[\pm 1]*T$ as illustrated below in Figure 2.1.5. A flype fixes the endpoints of the subtangle to which it is applied.

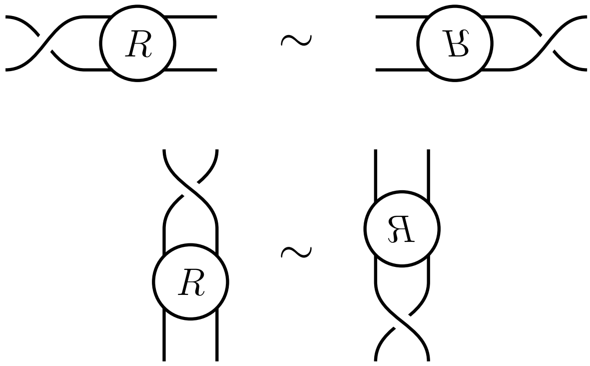


Figure 2.1.5: Flypes are a family of isotopies specific to 4 point tangles.

Flypes are important in the theory of alternating tangles.

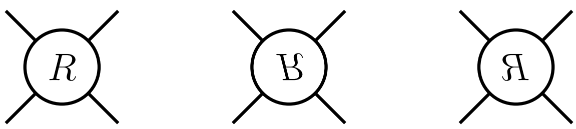
Definition 2.1.7 A tangle diagram is said to be *alternating* if the crossings encountered when following a component of the tangle around the diagram alternate from undercrossings to overcrossings. A tangle is *alternating* if it admits an alternating tangle diagram.

For example, in Figure 2.1.4 above, the two rightmost tangle diagrams are alternating while the first two diagrams are not alternating.

The proof of the classification theorem of rational tangles by the tangle fraction (Theorem 2.2.6) uses some heavy machinery involving flypes. The *Tait conjecture for knots*, proved by W. Menasco and M. Thistlethwaite in 1993 [MT93], states that two alternating knots are isotopic iff any two diagrams of the knots on S^2 are related by a finite sequence of flypes.

Using the Tait conjecture for knots, one proves the corresponding statement of the conjecture for rational tangles, thereby characterizing how alternating rational tangles are isotopic. Using this statement, one can then show that isotopic rational tangles have the same tangle fraction simply by checking that the tangle fraction is invariant under flypes. This task is not difficult once Proposition 2.1.9 below has been established.

Definition 2.1.8 The *horizontal flip* of a 4 point tangle T, denoted T^h, is obtained by rotating T by π through a horizontal line in the plane of T. Similarly the *vertical flip* of T, denoted T^v, is obtained via rotation of T by π through a vertical axis in the plane.



T $\quad$ T^h $\quad$ T^v

Flypes can be described more concretely in terms of these operations. Namely, a flype on a 4 point subtangle T is an isotopy of the form

$$[\pm 1] + T \sim T^h + [\pm 1] \quad \text{or} \quad [\pm 1] * T \sim T^v * [\pm 1].$$

Both horizontal and vertical flips are order 2 operations on 4 point tangles. Surprisingly, on rational tangles these operations have order 1.

Proposition 2.1.9 *If T is rational, then $T \sim T^h \sim T^v$.*

Proof: We prove that $T \sim T^h$; the other statement follows from this result by twisting T by $\pi/2$, locally applying the result, and twisting back. The proof is by induction.

The statement is clearly true for the $[0]$, $[\infty]$ and $[\pm 1]$ tangles. Assume the statement is true for a rational tangle with n crossings. Any rational tangle with $n+1$ crossings can be expressed as $[\pm 1] + T$, $T + [\pm 1]$, $[\pm 1] * T$ or $T * [\pm 1]$ where T is a rational tangle with n crossings.

The induction hypothesis easily gives

$$([\pm 1] + T)^h \sim [\pm 1] + T^h \sim [\pm 1] + T,$$

and similarly for $T + [\pm 1]$. Of the other two cases, we have

$$([\pm 1] * T)^h \sim T^h * [\pm 1] \sim T * [\pm 1] \sim [\pm 1] * T^v \sim [\pm 1] * T.$$

(in the third isotopy we applied a flype) and similarly $(T * [\pm 1])^h \sim T * [\pm 1]$. □

Corollary 2.1.10 *Rotation and inversion are operations of order 2 on rational tangles.*

Corollary 2.1.11 *For a rational tangle T,*

$$[\pm 1] + T \sim T + [\pm 1] \quad \text{and} \quad [\pm 1] * T \sim T * [\pm 1].$$

Proof: This is easy. With a flype, we have

$$[\pm 1] + T \sim T^h + [\pm 1] \sim T + [\pm 1].$$

The same argument shows multiplication by $[\pm 1]$ commutes. □

Note that this means every rational tangle can be constructed from the $[0]$ or $[\infty]$ tangles by a sequence of right additions or bottom multiplications of $[\pm 1]$: working from the outermost to innermost crossings of a tangle in twist form, if a crossing was added to the left or top, simply isotope it to the right or bottom respectively. We give a name to such rational tangles.

Definition 2.1.12 A rational tangle is said to be in *standard form* if it created from $[0]$ or $[\infty]$ by consecutive additions or products of $[\pm 1]$ on the right and bottom respectively. Every rational tangle is isotopic to a tangle in standard form.

(Note that as $[\infty] = ([0] + [1]) * [-1]$, we could actually remove $[\infty]$ in the definition above.)

Using the notation $[n] := \underbrace{[1] + \ldots + [1]}_{n \text{ times}}$ illustrated earlier in Figure 2.1.2, we can write a rational tangle in standard form as an expression

$$(\ldots((([a_1] * [\overline{a_2}]) + [a_3]) * [\overline{a_4}]) + \ldots * [\overline{a_{n-1}}]) + [a_n]$$

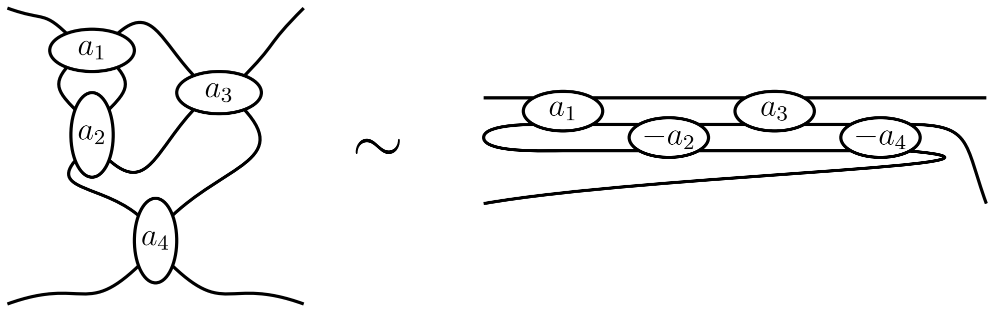


Figure 2.1.6: Although most rational tangles we consider will be in standard form (left), it is sometimes convenient to consider them as (partial closures) of elements of B_3.

where a_i are integers. We will abbreviate such an expression as $\langle a_1, \ldots, a_n \rangle$. Some illustrative examples of this notation are included below.

$\langle -2, 1, 2 \rangle$ $\langle 1, 1, 1, 1, 1 \rangle$

Most rational tangles we draw will be in standard form. But we can also consider rational tangles to be a partial closures of elements of B_3, the three-strand braid group, as illustrated in Figure 2.1.6. We will revisit this viewpoint in Section 5 when we look at how the reduced Burau representation of B_3 can be obtained from the Khovanov homology of rational tangles.

From this point on, we will assume all rational tangles are in standard form, unless otherwise stated.

2.2 The tangle fraction

We saw in the previous subsection that rational tangles are very structured. In particular, each admits a standard form (Definition 2.1.12). Because of their structure, rational tangles admit a nice classification. In this section we define the tangle fraction (Definition 2.2.4), and state the classification of rational tangles that uses it (Theorem 2.2.6). A consequence is that every rational tangle other than $[\infty]$ admits a unique canonical form (Corollary 2.2.9), a result crucial to Theorem 4.3.1, one of our main results.

The tangle fraction has several definitions[1], but we consider the one presented here to be the most intuitive. For the remainder of this section, it will often be convenient to denote the inverse of a tangle as $1/T$ instead of T^{-1}.

Lemma 2.2.1 *If a 4 point tangle T is rational, then*

$$T * [\bar{n}] = \frac{1}{[n] + \frac{1}{T}}, \qquad \textit{and} \qquad [\bar{n}] * T = \frac{1}{\frac{1}{T} + [n]}. \tag{2.2.1}$$

[1] The interested reader is referred to [KL03a] for alternative descriptions of the tangle fraction.

Proof: Since
$$(T * [\overline{n}])^r = T^r + [-n],$$
we have
$$(T * [\overline{n}])^{-1} = -(T^r + [-n]) = T^{-1} + [n].$$
The results follow. $\square$

Since every rational tangle is isotopic to a rational tangle in standard form
$$\langle a_1, \ldots, a_n \rangle = (\ldots((([a_1] * [\bar{a}_2]) + [a_3]) * [\bar{a}_4]) + \ldots * [a_{n-1}]) + [a_n] \ ,$$
the lemma allows us to write a rational tangle as 'continued fraction' of tangles.

Definition 2.2.2 A rational tangle is a *continued fraction in integer tangles* if it has the form
$$[a_1, a_2, \ldots, a_n] := [a_1] + \cfrac{1}{[a_2] + \frac{1}{[a_3] + \ldots + \frac{1}{[a_{n-1}] + \frac{1}{[a_n]}}}}.$$
We stipulate that $a_2, \ldots, a_n \in \mathbb{Z} \setminus \{0\}$ and $a_1 \in \mathbb{Z}$. If $a_1 = 0$ in a continued fraction of integer tangles, when we write the expression we omit the $[0]$ term.

Rational tangles in standard form can be converted to a continued fraction in integer tangles by multiple applications of the relation $[n] + T \sim T + [n]$ and Lemma 2.2.1. The interconversion can be summarized by
$$\langle a_1, a_2, \ldots, a_n \rangle \sim [a_n, a_{n-1}, \ldots, a_1].$$

We now list some properties of rational tangles in continued fraction form. The proof of each is not difficult.

Lemma 2.2.3 *Let $T = [a_1, \ldots, a_n]$ be a rational tangle in continued fraction form. Then*

1. $T + [\pm 1] = [a_1 \pm 1, a_2, \ldots, a_n]$,
2. $\frac{1}{T} = [0, a_1, a_2, \ldots, a_n]$,
3. $-T = [-a_1, -a_2, \ldots, -a_n]$.

We now come to the main definition of this subsection.

Definition 2.2.4 Let $T = [a_1, \ldots, a_n]$ be a rational tangle in continued fraction form. Define the *fraction* or *tangle fraction* $F(T)$ of T to be the rational number obtained by replacing the integer tangles in the expression for T with their corresponding integers. That is,
$$F(T) = a_1 + \cfrac{1}{a_2 + \frac{1}{a_3 + \ldots + \frac{1}{a_{n-1} + \frac{1}{a_n}}}}. \tag{2.2.2}$$
If $T = [\infty]$ we define $F([\infty]) = \infty$.

Although the tangle fraction is well-defined for rational tangles in a continued fraction form, it is not clear if this is well-defined for rational tangles up to isotopy. That is, is isotopic rational tangles expressed as different continued fractions have the same tangle fractions. This is one of the directions of the main theorem of this section (Theorem 2.2.6). Namely, if $S \sim T$, then $F(S) = F(T)$ for any choice of a continued fraction of tangles for S and T.

We now list some properties of the tangle fraction. They all follow easily from Lemma 2.2.3.

Lemma 2.2.5 *Let $T = [a_1, \ldots, a_n]$ be as above. Then*

1. $F(T + [\pm 1]) = F(T) \pm 1$,
2. $F(\frac{1}{T}) = \frac{1}{F(T)}$,
3. $F(T * [\,\overline{n}\,]) = \frac{1}{[n] + \frac{1}{F(T)}}$,
4. $F(-T) = -F(T)$.

We are now in a position to state the main result of this section, though omit its proof. (The proof is given in [KL03b]. Though it is technical, it uses no other tools than those previously described.)

Theorem 2.2.6 *The tangle fraction is well-defined up to isotopy, and two rational tangles are isotopic iff they have the same fraction.*

Example 2.2.7 Consider the rational tangles below. The first has standard form $\langle -3, 1, 1\rangle$ with the second is $2 + [\bar{2}]$, which has continued fraction form $[2, 2]$.

After computing their fractions,

$$F(\langle -3, 1, 1\rangle) = 1 + \frac{1}{1 + \frac{1}{-3}} = 1 + \frac{3}{2} = 2 + \frac{1}{2} = F([2, 2]),$$

we find the tangles are isotopic. This may not be immediately evident from inspection.

Definition 2.2.8 A continued fraction

$$a_1 + \frac{1}{a_2 + \ldots + \frac{1}{a_{n-1} + \frac{1}{a_n}}}$$

is said to be in *canonical form* if n is odd and either $a_1 \geq 0$ and $a_i > 0$, or $a_1 \leq 0$ and $a_i < 0$ (for $i \neq 1$). The same definition extends to rational tangles, provided they are in continued fraction form.

Corollary 2.2.9 *Every rational tangle other than $[\infty]$ has a unique canonical form.*

Proof: Consider an arbitrary rational tangle T. Without loss of generality assume $F(T) > 0$. By Euclid's algorithm we can uniquely write $F(T)$ as a continued fraction

$$a_1 + \frac{1}{a_2 + \ldots + \frac{1}{a_{n-1} + \frac{1}{a_n}}}$$

where $a_1 \geq 0$ and $a_i > 0$ for all $1 < i < n$, and $a_n > 1$. If n is even, we replace a_n with $(a_n - 1) + \frac{1}{1}$. Now $F(T)$ is in canonical form, and is unique by construction. The claim follows from Theorem 2.2.6 by considering the corresponding continued fraction of tangles. □

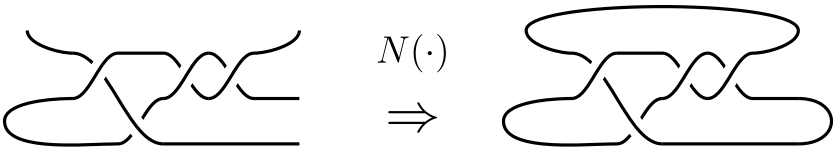


Figure 2.3.1: The figure-8 knot, like any rational link, is alternating.

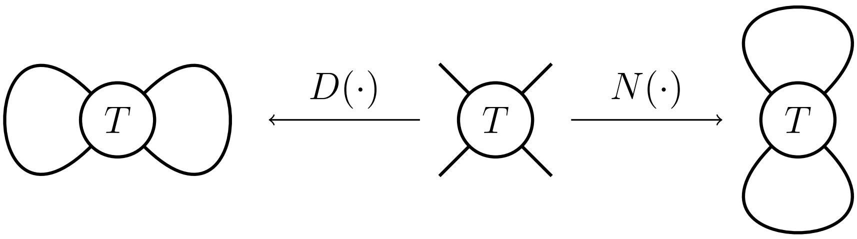
Note that this implies that every rational tangle admits a diagram whose crossings are all of the same type. (That is, the gradients of the overcrossing lines all agree.) This follows by considering the standard form $\langle a_1, \ldots, a_n \rangle$ of a tangle in canonical form. If $a_i \geq 0$ or $a_i \leq 0$ for all i then by definition of the standard form of a rational tangle, every crossing has the same configuration. We will use this fact in Section 4 to inductively construct the Khovanov homology of a rational tangle.

We have now seen the main components and highlights to the theory of rational tangles. Strictly speaking, we've seen all we need for the later sections, but it'd be a shame to skip some of the nice results regarding rational links that the theory of rational tangles provides. As such, before we begin the next topic, Khovanov homology, we briefly pause to look at links obtained from rational tangles.

2.3 Rational links

Given a rational tangle, one can close the ends in several ways to obtain a *rational link*. Two of these, the *numerator closure* $N(\cdot)$ and *denominator closure* $D(\cdot)$ are illustrated below. The Hopf link, for example, can be expressed as $N([2])$. The links we consider in examples in the sequel will be rational.

Sometimes both types of closure give isotopic links; trivially $N([1])$ and $D([1])$ are both isotopic to the unknot. Sometimes non-isotopic tangles close to give the same link; trivially $N([\overline{n}])$ is isotopic to the unknot for any n. Since $D(T) = N(T^r)$, we will only consider the numerator closure of a rational tangle in the sequel.



Given a rational tangle, it may not be immediately clear if its numerator closure will yield a link with one or two components. The distinction is trivial using the tangle fraction [KL03a].

Proposition 2.3.1 *Let $N(p/q)$ denote the rational link obtained by taking the numerator closure of the rational tangle with fraction p/q, where p and q are relatively prime. Then $N(p/q)$ has two connected components iff p is even and q is odd. That is, iff p/q has parity e/o.*

Given that the tangle fraction classifies rational tangles, it is perhaps unsurprising that it classifies rational knots too. The following theorem is due to Schubert [Sch56].

Theorem 2.3.2 *Rational links $N(p/q)$ and $N(p'/q')$ are isotopic iff:*

1. $p = p'$, *and*
2. *either* $q \equiv q'$ *mod* p *or* $qq' \equiv 1$ *mod* p.

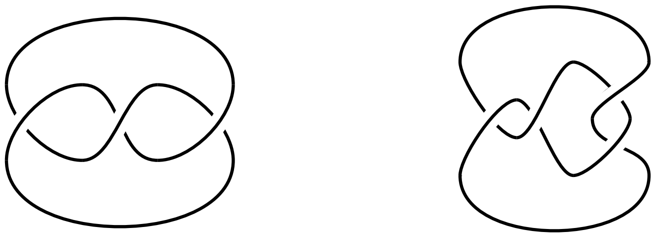
Schubert originally stated this classification of rational links in terms of 2-bridge links as his work predates Conway's theory of rational tangles. A combinatorial proof of the theorem that uses the classification of rational tangles appears in [KL03a]. An oriented version of the theorem also exists and is due to Schubert too; its phrasing and proof can be found in [KL03a] as well.

A consequence of Theorem 2.3.2 is that chiral rational links admit an easy classification.

Definition 2.3.3 A link L is *chiral* if it is isotopic to its mirror image. That is, if $L \sim -L$. A link is *achiral* if it is not chiral.

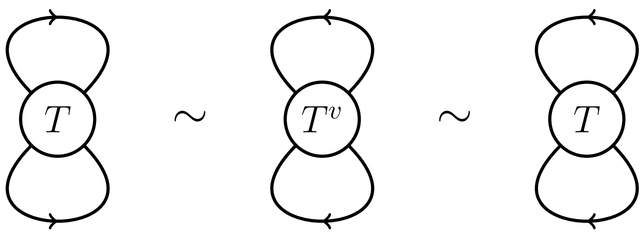
Proposition 2.3.4 *Let $L = N(p/q)$ as above. Then L is achiral iff $q^2 \equiv -1$ mod p.*

Example 2.3.5 It is well-known that the trefoil and figure-8 knot are chiral and achiral respectively, these properties follow trivially from the previous proposition. The knots can be realized as $N([3])$ and $N([2,2])$, illustrated below. Since $F([3]) = 3/1$ and $1 \not\equiv -1$ mod 3, the trefoil is chiral. As $F([2,2]) = 5/2$, and $4 \equiv -1$ mod 5, the figure-8 knot is achiral.



Definition 2.3.6 Given an oriented link K, the *inverse* of K, denoted K^*, is obtained by reversing the orientation of each component. A link is *invertible* if it is isotopic to its inverse, that is, if $K \sim K^*$.

It is an easy consequence of Proposition 2.1.9 that every rational link is invertible. The proof follows by applying a vertical flip to the knot, as illustrated below.



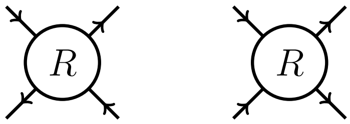
A stronger notion of invertibility can be defined for links with more than one component.

Definition 2.3.7 A link L is said to be *strongly invertible* if any link formed by reversing the orientations of any of the components of L is isotopic to L.

Strongly invertible rational links admit a classification using the oriented version of the classification theorem for oriented rational links [KL03b].

Proposition 2.3.8 *Let L be a two-component oriented link with $L = N(p/q)$ as above, with $|p| > |q|$. Then L is strongly invertible iff $q^2 = 1 + op$ for some odd integer o.*

Finally, for notational purposes in Section 4, we note that oriented rational tangles that have well-defined numerator closures are isotopic to one of two forms, illustrated below. We refer to these as *type I* and *type II* (oriented) rational tangles.



I $\quad\quad$ II

This concludes our foray into rational links.

3 Khovanov homology

In this section we review Bar-Natan's dotted Khovanov homology theory of tangles, which we use to study the Khovanov homology of rational tangles in Section 4.

In Section 3.1 we construct Bar-Natan's extension of Khovanov homology to tangles, as described in [Bar04].

In Section 3.2 we modify the Khovanov homology theory developed in the previous subsection to a 'dotted' version. In the dotted version complexes are far easier to manipulate, and when restricted to links the theory is equivalent to ordinary Khovanov homology. The dotted theory is described in [Bar06].

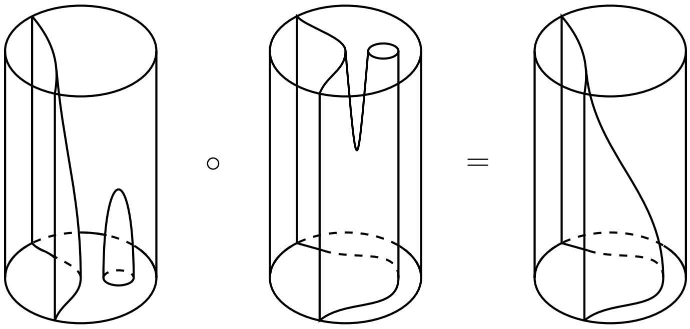
Finally, in Section 3.3 we prove the Khovanov bracket is invariant under the Reidemeister moves, following the proof as hinted by Bar-Natan in [Bar06]. The proof demonstrates how the tools described in the previous subsection can be applied.

3.1 Bar-Natan's approach

In an effort to try and keep this paper as self-contained as possible, we present Bar-Natan's extension to Khovanov homology to tangles. His exposition is so clear and self-contained that we think it would be a mistake not to follow it. As such, we embarrassingly present a 'compressed' version of the exposition from [Bar04] we need. If at any stage the reader is lost by what we write below, we refer her directly to the source.

Definition 3.1.1 The category $\mathcal{C}ob^3(\emptyset)$ is defined as follows. Its objects are smoothings (simple curves in the plane) with Hom-sets comprised of cobordisms between such smoothings. By a cobordism we mean an oriented two-dimensional surface embedded into $\mathbb{R}^2 \times [0, 1]$ whose boundary lies in $\mathbb{R}^2 \times \{0, 1\}$. More generally, for a finite set of points B in S^1, define a category $\mathcal{C}ob^3(B)$ as follows. Its objects are smoothings but now with boundary B. The Hom-sets $\mathcal{C}ob^3(B)(\mathcal{O} \to \mathcal{O}')$ between two smoothings in $\mathcal{C}ob^3(B)$ consist of all oriented two-dimensional surfaces embedded into a cylinder with boundary $(\mathcal{O} \times \{0\}) \cup (B \times [0, 1]) \cup (\mathcal{O}' \times \{1\})$. In both categories we consider cobordisms up to boundary preserving isotopies. Composition is defined by placing one cobordism on top of the other and vertically renormalizing the result.

The term $\mathcal{C}ob^3$ will sometimes be used as a generic reference to $\mathcal{C}ob^3(\emptyset)$ or $\mathcal{C}ob^3(B)$.



We now create a category consisting of vectors and matrices in $\mathcal{C}ob^3$. Recall that a category is *preadditive* if its Hom-sets are Abelian groups and composition of morphisms is bilinear. Note that every category $\mathcal{C}$, if not already preadditive, can be modified to a preadditive category $\mathcal{C}'$ by defining the Hom-sets $\mathcal{C}'(\mathcal{O} \to \mathcal{O}')$ to be the free Abelian group generated by $\mathcal{C}(\mathcal{O} \to \mathcal{O}')$.

Definition 3.1.2 Given a preadditive category $\mathcal{C}$, define a preadditive category $\mathrm{Mat}(\mathcal{C})$ as follows. The objects of $\mathrm{Mat}(\mathcal{C})$ are formal direct sums (possibly empty) of objects of $\mathcal{C}$. The Hom-sets $\mathrm{Mat}(\mathcal{C})(\oplus_{i=1}^m \mathcal{O}_i \to \oplus_{j=1}^n \mathcal{O}'_j)$ consist of $m \times n$ matrices (F_{ij}) of morphisms $F_{ij} : \mathcal{O}_i \to \mathcal{O}'_j$ in $\mathcal{C}$. Addition in $\mathrm{Mat}(\mathcal{C})(\oplus_{i=1}^m \mathcal{O}_i \to \oplus_{j=1}^n \mathcal{O}'_j)$ is just matrix addition while composition of morphisms in $\mathrm{Mat}(\mathcal{C})$ is defined in the obvious way:

$$(F \circ G)_{ik} := \sum_j F_{ij} \circ G_{jk}.$$

We will sometimes represent the objects of $\mathrm{Mat}(\mathcal{C})$ by column vectors and the morphisms of $\mathrm{Mat}(\mathcal{C})$ as (marked) arrows taking one column to another. (Such as in the proof of the invariance of the Khovanov bracket under R2 and R3 in Section 3.3.)

Definition 3.1.3 Given a preadditive category $\mathcal{C}$, define the category of complexes $\mathrm{Kom}(\mathcal{C})$ to be the category with objects chains of finite length

$$\cdots \xrightarrow{\partial_{n-1}} C_n \xrightarrow{\partial_n} C_{n+1} \xrightarrow{\partial_{n+1}} C_{n+2} \xrightarrow{\partial_{n+2}} \cdots$$

(only finitely many terms are not the zero object) such that $\partial_n \circ \partial_{n-1} = 0$ for all n.

Define the Hom-sets $\mathrm{Kom}(\mathcal{C})(C \to D)$ to be the set of all chain maps from C to D. Chain maps $F : C \to D$ are, as usual, collections of morphisms $(F_i)_{i \in \mathbb{Z}}$ taking $F_i : C_i \to D_i$ and satisfying $\partial_n F_n = F_{n-1} \partial_n$. Composition in $\mathrm{Kom}(\mathcal{C})$ is defined via $(F \circ G)_r = F_r \circ G_r$.

We now have the appropriate definitions to define the Khovanov complex $[\![T]\!]$ of an oriented tangle diagram T.

Definition 3.1.4 Let n_+ and n_- be the number of positive and negative crossings in an oriented tangle diagram T, and the total number of crossings be n. Number the crossings from 1 to n. Let S be the set of states (possible smoothings) for T, expressed as a word in $\{0, 1\}$ where the n-th letter denotes the smoothing assigned to the n-th crossing. (That is, S is the set of all words of length n in $\{0, 1\}$.)

Construct a n-dimensional 'combinatorial' cube from S as follows. The vertices are the elements of S, and two vertices are connected by an edge iff the vertices differ from

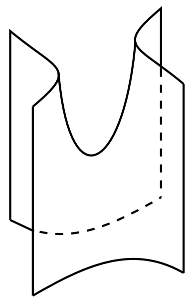
each other by one letter. Orient these edges, representing them as arrows, so that the head of the arrow points towards the larger word (when the words are viewed as integers).

We also attach a word of length n in $\{0, 1, \star\}$ to each arrow. The word is identical in each letter to the words on the head and tail of the arrow, except in the place where they differ, for which the word has letter $\star$.

To finish constructing the combinatorial cube, assign a positive or negative sign to each arrow depending the word attached to it: if an arrow is labeled by the word $\xi = \xi_1\xi_2\cdots\xi_n$, then the arrow has sign $(-1)^{\sum_{i<j}\xi_i}$ where $\xi_j = \star$.

The combinatorial cube corresponding to a diagram with 2 crossings is pictured below.

$$\begin{tikzcd}
 & 01 \arrow[dr, "\star 1"] & \\
00 \arrow[ur, "0\star"] \arrow[dr, "\star 0"'] & & 11 \\
 & 10 \arrow[ur, "-1\star"'] &
\end{tikzcd}$$



Now turn the combinatorial cube into a 'topological' cube. Replace each vertex V with the smoothing of T associated to the state V. Leave the arrows, but replace the word ξ on each arrow by a cobordism M in $\mathcal{C}ob^3(\partial T)(\xi(0) \to \xi(1))$ constructed as follows. (Here $\xi(i)$ is the smoothing of T associated to the state obtained from ξ by replacing $\star$ with i.)

Aside from a small disk D containing the smoothings of the crossing corresponding to $\star$, the smoothings $\xi(0)$ and $\xi(1)$ are identical. Define M to be equal to the identity morphism outside of $D \times [0, 1]$ — that is, equal to $(D^c \cap T) \times [0, 1]$. To complete M, fill in the missing cylindrical slot with a saddle cobordism, illustrated above.

To complete the construction of the topological cube, view the cobordism as an element in the preadditive category $\mathcal{C}ob^3(\partial T)$, making it positive or negative depending on the sign associated to the arrow ξ is on.

We have constructed a topological cube whose vertices are objects of $\mathcal{C}ob^3(\partial T)$ and whose edges are morphisms in $\mathcal{C}ob^3(\partial T)$.

We now view the cube as an element of $\mathrm{Kom}(\mathrm{Mat}(\mathcal{C}ob^3(\partial T)))$. To achieve this, first consider the set of states S, viewed as words in $\{0, 1\}$. Partition S into $n + 1$ sets by the number of 0s in a word. Now turn each set into a vector, arranging the words from smallest to largest (when viewed as integers). Refer to the vector consisting of words with i 0s as the i-th vector.

This series of vectors of words in $\{0, 1\}$ encode a way to arrange the vertices of the topological cube into a series of vectors of smoothings in the obvious way. Once the vertices of the topological cube have been arranged into these vectors, each vector of smoothings can be regarded as an object of $\mathrm{Mat}(\mathcal{C}ob^3(\partial T))$. The set of arrows between the vectors can be regarded as a matrix in $\mathrm{Mat}(\mathcal{C}ob^3(\partial T))$. (If there is no arrow between entries in the vectors, the map is a zero morphism.)

Finally, in order to view the topological cube as a complex in $\mathrm{Mat}(\mathcal{C}ob^3)$, we must assign homological degrees to each vector. Assign to the n-th vector the homological degree $n - n_-$. This complex is the *Khovanov complex of* T, denoted $[\![T]\!]$. We call $[\![\,\cdot\,]\!]$ the *Khovanov bracket*.

We need to check that the Khovanov bracket is well-defined. That is, that $[\![T]\!]$ is actually a complex for all T.

Proposition 3.1.5 *For any oriented tangle diagram T, $[\![T]\!] \in \mathrm{Kom}(\mathrm{Mat}(\mathcal{C}ob^3(\partial T)))$.*

Proof: The claim will follow if we can show that every face of the topological cube anticommutes. To see why, note that the entry $(\partial_{n+1} \circ \partial_n)_{ij}$ is equal to the sum of all paths in the cube from $[\![T]\!]^n_i$ to $[\![T]\!]^{n+2}_j$. (Where $[\![T]\!]^n_i$ is the ith entry in the vector $[\![T]\!]^n$.) These paths are precisely the paths on the face of the topological cube containing $[\![T]\!]^n_i$ and $[\![T]\!]^{n+2}_j$. It follows that if every face anticommutes, every entry of $\partial_{n-1} \circ \partial_n$ will be zero.

Note that if we ignore signs on the cobordisms, each face in the topological cube commutes. This is because we are considering cobordisms up to isotopy, and spatially separated saddles in a cobordism can be time-reordered by isotopy.

The anticommutivity of the faces then follows if we can show each face has an odd number of negative cobordisms. Consider a face of the combinatorial cube, without signs, illustrated below. Here the words w and w''' differ in the j-th and k-th letters. Letters that have changed are indicated with a tilde.

$$
\begin{tikzcd}[row sep=small]
w_1\cdots\star\cdots w_k\cdots w_n & & w' \arrow[dr] & w_1\cdots\tilde{w}_j\cdots\star\cdots w_n \\
 & w \arrow[ur] \arrow[dr] & & w''' \\
w_1\cdots w_j\cdots\star\cdots w_n & & w'' \arrow[ur] & w_1\cdots\star\cdots\tilde{w}_k\cdots w_n
\end{tikzcd}
$$

Recalling the definition of the combinatorial cube, one can see that arrows from $w \longrightarrow w'$ and $w'' \longrightarrow w'''$ will have the same sign, while the sign of $w' \longrightarrow w'''$ and $w \longrightarrow w'''$ will differ. It follows that each face in the topological cube has an odd number of negative cobordisms. $\square$

We now have a way of associating a complex to a diagram of a tangle, however the Khovanov bracket is not currently a tangle invariant. When considered up to homotopy equivalence in a certain quotient category $\mathrm{Kom}(\mathrm{Mat}(\mathcal{C}ob^3_{/l}))$, however, it is.

Definition 3.1.6 Two morphisms $F, G : C \to D$ in $\mathrm{Kom}(\mathcal{C})$ are said to be *homotopic*, denoted $F \sim G$, if there exist morphisms $(h_i)_{i\in\mathbb{Z}}$ with $h_i : C_i \to D_{i-1}$ for which $F_i - G_i = \partial_{i+1} h_i + h_{i-1}\partial_i$.

We note that this notion of homotopy is no different to that in usual homological algebra, except we are now in a more complicated category than vector spaces or modules. It is not difficult to verify that homotopy is an equivalence relation, and is invariant under left and right composition. We write $\mathrm{Kom}_{/h}(\mathcal{C})$ to denote $\mathrm{Kom}(\mathcal{C})$ modulo homotopies. In order to simplify notation, we will denote $\mathrm{Kom}(\mathrm{Mat}(\mathcal{C}ob^3_{/l}(\partial T)))$ and $\mathrm{Kom}(\mathrm{Mat}(\mathcal{C}ob^3_{/l}(\emptyset)))$ by $\mathrm{Kob}(\partial T)$ and $\mathrm{Kob}(\emptyset)$. We will sometimes refer to both categories collectively as Kob.

Definition 3.1.7 Two complexes C, D are said to be *homotopy equivalent*, denoted $C \sim D$, if they are isomorphic in $\mathrm{Kom}_{/h}(\mathcal{C})$. That is, if there are chain maps $F : C \to D$ and $G : D \to C$ in $\mathrm{Kom}(\mathcal{C})$ for which $G \circ F$ and $F \circ G$ are homotopic to the identity.

It is easy to check that homotopy equivalence is an equivalence relation on complexes. We now define the quotient category for which the Khovanov bracket is a tangle invariant.

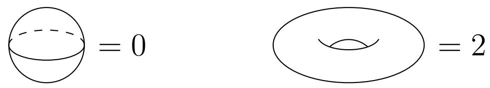
Definition 3.1.8 Let $\mathcal{C}ob^3_{/l}$ be the quotient category obtained from $\mathcal{C}ob^3$ by reducing $\mathcal{C}ob^3$ by the local relations S, T and *4Tu* described below.

S: Whenever a cobordism contains a component isotopic to a sphere, the cobordism is equal to 0.

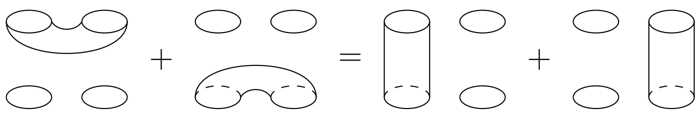
T: Whenever a cobordism contains a component isotopic to a torus, the component may be removed and the remaining cobordism multiplied by a factor of 2.

$4Tu$: Assume the intersection of a cobordism C with a ball is the union of four disks D_1 through to D_4. Let C_{ij} denote the result of removing D_i and D_j from C and replacing them by a tube with the same boundary. The $4Tu$ relation states $C_{12} + C_{34} = C_{13} + C_{24}$.

We can summarize the above relations with the following pictures.



S T



$4Tu$

We are now in a position to state exactly how the Khovanov bracket is a tangle invariant.

Theorem 3.1.9 *The isomorphism class of* $[\![T]\!]$ *regarded in* $\mathrm{Kob}_{h/}(\partial T)$ *is an invariant of the tangle* T. *That is,* $[\![\cdot]\!]$ *is invariant under the Reidemeister moves. When* $[\![T]\!]$ *is constructed, different orderings of the crossings chosen in its construction yield isomorphic complexes.*

We will not prove that this version of the Khovanov bracket is invariant under the Reidemeister moves. (But we will prove the 'dotted' version of Bar-Natan's theory we use in Section 4 is invariant under the Reidemeister moves in Subsection 3.3.) The proof of the invariance of the bracket in [Bar04] involves the construction of explicit chain homotopies, and is in many respects one of the hardest sections of the paper. We now discuss the excellent composition properties the Khovanov bracket enjoys.

Definition 3.1.10 A d-input *planar arc diagram* D is an 'output' disk with:

- d smaller 'input' disks removed. The input disks are numbered 1 to d, each with a basepoint $(*)$ on its boundary.
- a collection of disjoint embedded arcs that are closed or begin and end on the boundary of an output or input disk transversely.

The output disk contains a basepoint on its boundary too. We consider planar arc diagrams up to isotopy, and they may be oriented or unoriented.

An example of a planar arc diagram can be found in Example 3.1.17. In the sequel, we will drop the basepoints from planar arc diagrams.

Definition 3.1.11 A collection of sets $\mathcal{P}(k)$ along with operations defined for each (oriented) unoriented planar arc diagram is an *(oriented) planar algebra* if radial planar arc diagrams act as identities and the following associativity condition holds: if D_i is the result of placing D' into the i-th hole of D (providing the diagrams are compatible), then as operations $D_i = D \circ (I \times \cdots \times D' \times \cdots I)$.

The lack of restrictions on the operations defined for each planar arc diagram makes the definition quite general, though we usually consider sets comprised of topological objects.

Example 3.1.12 Let $\mathcal{T}^0(k)$ denote the collection of all unoriented tangle diagrams in a based disk (a disk with a basepoint on its boundary) with k ends on the boundary, considered up to planar isotopies as usual. Let $\mathcal{T}(k)$ denote the quotient of $\mathcal{T}^0(k)$ by the Reidemeister moves.

Then $(\mathcal{T}^0(k))_k$ and $(\mathcal{T}(k))_k$ are planar algebras. Let D be a d-input planar arc diagram D with k_i arcs ending on the i-th input disk and k arcs on output disk. Then define an operator

$$D : \mathcal{T}^0(k_1) \times \cdots \times \mathcal{T}^0(k_d) \longrightarrow \mathcal{T}^0(k)$$

by placing the d input tangles into the holes of D. The operators are similarly defined for $(\mathcal{T}(k))_k$. It is clear that the radial planar arc diagrams act as the identity operators. The associativity condition is easily seen to hold too.

Example 3.1.13 The collection $\mathrm{Obj}(\mathcal{C}ob^3_{/l})$ and the Hom-sets $\mathrm{Mor}(\mathcal{C}ob^3_{/l})$ are both planar algebras. The first is just a sub planar algebra of $\mathcal{T}(k)$. For the second, we need to specify how a planar arc diagram can act as an operator on cobordisms. The solution is not difficult – $D \times [0,1]$ is a vertical cylinder with d vertical cylindrical holes connected by vertical curtains. One simply places cobordisms in the holes, which defines an operation from $D : (\mathrm{Mor}(\mathcal{C}ob^3_{/l}))^d \to \mathrm{Mor}(\mathcal{C}ob^3_{/l})$.

Both the previous examples may seem, for lack of a better word, obvious: clearly one can fill up the holes of a planar arc diagram with tangles to obtain more tangles.

There are situations where the operator a planar arc diagram should correspond to is not as clear. For instance, say we decided to put complexes of tangles in the holes of a planar arc diagram. That is, put elements of Kob into planar arc diagrams. Can the planar arc diagrams act as operators on these complexes?

The answer is yes, and the operation is essentially the notion of a tensor product of complexes, with the planar arc diagram gluing together the pieces. Let us formalize this. Consider $\mathrm{Kob}(B_k)$ and $\mathrm{Kob}_{/h}(B_k)$ where where B_k is some placement of k points on a based circle.

Given a d-input planar arc diagram D with k_i arcs ending on the i-th input disk and k arcs ending on the outer boundary, D acts as an operator as follows.

Given complexes $(\Omega_i, d_i) \in \mathrm{Kob}(B_{k_i})$, define $D(\Omega_1 \ldots, \Omega_d) = (\Omega, d)$ by

$$\Omega^r := \bigoplus_{r=r_1+\ldots r_d} D(\Omega_1^{r_1}, \ldots, \Omega_d^{r_d}), \tag{3.1.1}$$

$$d_{|D(\Omega_1^{r_1},\ldots,\Omega_d^{r_d})} := \sum_{i=1}^{d} (-1)^{\sum_{j<1} r_j} D(I_{\Omega_1^{r_1}}, \ldots, d_i, \ldots, I_{\Omega_d^{r_d}}). \tag{3.1.2}$$

The basic properties of tensor products transfer over to tensor products in this context; in particular a morphism $\Psi_i : \Omega_{ia} \to \Omega_{ib}$ induces a morphism $D(I, \ldots, \Psi_i, \ldots, I) : D(\Omega_1, \ldots, \Omega_{ia}, \ldots, \Omega_d) \to D(\Omega_1, \ldots, \Omega_{ib}, \ldots, \Omega_d)$. Homotopies at the level of tensor factors induce homotopies at the levels of tensor products.

These facts essentially constitute the proof of the following proposition.

Proposition 3.1.14 *The collection* $(\mathrm{Kob}(B_k))$ *is a planar algebra. Furthermore, the operations* D *on* $(\mathrm{Kob}(B_k))$ *send homotopy equivalent complexes to homotopy equivalent complexes, meaning* $\mathrm{Kob}_{/h}(B_k)$ *also has a natural structure of a planar algebra.*

The upshot of all this is that the Khovanov bracket respects the tensoring operations associated to a planar arc diagram. We need another definition to state this precisely.

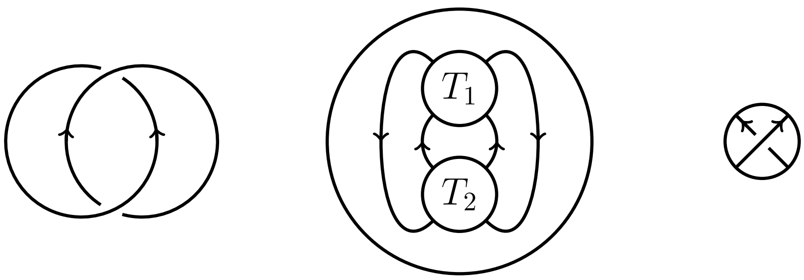
Definition 3.1.15 A *morphism* Φ *of planar algebras* from $(\mathcal{P}(k)) \to (\mathcal{Q}(k))$ is a collection of maps $\Phi : \mathcal{P}(k) \to \mathcal{Q}(k)$ satisfying $\Phi \circ D = D \circ (\Phi \times \cdots \times \Phi)$ for every D.

Theorem 3.1.16 *The Khovanov bracket* $[\![\cdot]\!]$ *is an oriented planar algebra morphism* $[\![\cdot]\!] : (\mathcal{T}(s)) \to (\mathrm{Kob}_{/h}(B_s))$.

One proves this by showing that the theorem holds when all inputs to a planar arc diagram are restricted to single crossings; the result in general follows from this and the associativity of the planar algebras involved.

We will use Theorem 3.1.16 extensively in the sequel. Before we develop the theory further, we pause to go through an example utilizing these concepts.

Example 3.1.17 Let us calculate the Khovanov complex of the Hopf link below. We can view the link as the result of inserting the tangle diagrams below into the planar arc diagram illustrated.



$D(X_1, X_2)$ D $X_1 = X_2$

We can then use the property that $[\![\cdot]\!]$ is a morphism of planar algebras:

$$[\![\,\mathrm{Hopf}\,]\!] = [\![D(X_1, X_2)]\!] = D([\![X_1]\!], [\![X_2]\!]).$$

From the definition of the Khovanov bracket we have

$$[\![X_1]\!] \quad = \quad [\![X_2]\!] \quad = \quad \underline{)\,(} \xrightarrow{\;)\!-\!(\;} \asymp \quad .$$

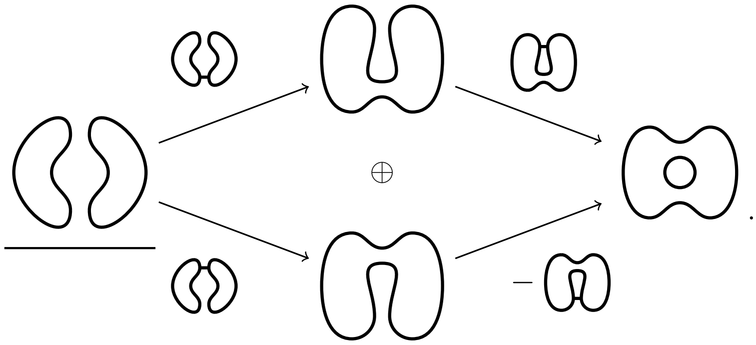
Here, and in the sequel, we've underlined the object of the complex that is in homological degree zero. The symbol $)\!-\!($ is a shorthand to denote the saddle map; in depictions of saddle cobordisms in the sequel we will use the same notation.

Using equations (3.1.1) and (3.1.2), we construct $[\![\,\mathrm{Hopf}\,]\!]$.

$$\begin{tikzcd}
 & D()(, \asymp) \arrow[dr, "D()\!-\!(, 1)"] & \\
\underline{D()(,)()} \arrow[ur, "D(1,)\!-\!()"] \arrow[dr, "D()\!-\!(, 1)"'] & \oplus & D(\asymp, \asymp) \\
 & D(\asymp,)() \arrow[ur, "-D(1,)\!-\!()"'] &
\end{tikzcd}$$

$$[\![\,\mathrm{Hopf}\,]\!]^0 \longrightarrow [\![\,\mathrm{Hopf}\,]\!]^1 \longrightarrow [\![\,\mathrm{Hopf}\,]\!]^2$$

When we calculate the objects and morphisms of this complex by placing the appropriate smoothings and cobordisms into D, we obtain



The original Khovanov homology theory was bigraded. We now equip the theory developed so far with a bigrading too.

Definition 3.1.18 A *graded category* is a preadditive category $\mathcal{C}$ with the following two properties:

1. The Hom-sets of $\mathcal{C}$ form a graded abelian group, composition respects the gradings ($\deg f \circ g = \deg f + \deg g$), and all identity maps have degree 0.
2. There is a $\mathbb{Z}$-action $(m, \mathcal{O}) \mapsto \mathcal{O}\{m\}$, called "grading shift by m" on the objects of $\mathcal{C}$. As plain abelian groups, morphisms are unchanged by the action, $\mathrm{Mor}(\mathcal{O}_1\{m_1\}, \mathcal{O}_2\{m_2\}) = \mathrm{Mor}(\mathcal{O}_1, \mathcal{O}_2)$, but gradings do: if $f \in \mathrm{Mor}(\mathcal{O}_1, \mathcal{O}_2)$ and $\deg f = d$, then as an element of $\mathrm{Mor}(\mathcal{O}_1\{m_1\}, \mathcal{O}_2\{m_2\})$ we have $\deg f = d + m_2 - m_1$.

If a preadditive category $\mathcal{C}$ satisfies only the first property above, it can be extended to a category $\mathcal{C}'$ that has the second property too: let the objects of $\mathcal{C}'$ be copies of the original objects but with integers attached. (That is, $\mathcal{O}\{m\}$ for all $\mathcal{O} \in \mathcal{C}$ and $m \in \mathbb{Z}$.) It is clear how to define a $\mathbb{Z}$-action on $\mathcal{C}'$, how to define the Hom-sets $\mathrm{Mor}\,\mathcal{C}'$, and how to grade $\mathrm{Mor}\,\mathcal{C}'$.

If we extend a preadditive category $\mathcal{C}$ to a graded category we will still refer to the graded version as $\mathcal{C}$. When writing a complex in a graded category, we will often denote the gradings of its objects in brackets. (Such as in the proof of the invariance of the Khovanov bracket under R1 in Section 3.3.)

We note that if $\mathcal{C}$ is graded category then $\mathrm{Mat}(\mathcal{C})$ can be considered as a graded category too: a matrix is homogeneous in degree d iff all its entries are of degree d. Similarly complexes in $\mathrm{Kom}(\mathcal{C})$ (or $\mathrm{Kom}(\mathrm{Mat}(\mathcal{C}))$) are graded categories.

We now define a grading on the Hom-sets of $\mathcal{C}ob^3$, and as a result Kob becomes a graded category.

Proposition 3.1.19 *For a cobordism $C \in \mathrm{Mor}(\mathcal{C}ob^3(B))$ with $|B|$ vertical boundary components, define $\deg C := \chi(C) - \frac{1}{2}|B|$, where χ is the Euler characteristic. Then $\mathcal{C}ob^3(B)$, and $\mathcal{C}ob^3_{/l}(B)$ are graded categories.*

Proof: It is not hard to verify that the degree is additive under composition in $\mathcal{C}ob^3(B)$, and under horizontal composition using the planar algebra structure of $\mathcal{C}ob^3(B)$. It is a simple matter to verify that the S, T and *4Tu* relations are degree-homogeneous, from which it follows that $\mathcal{C}ob^3_{/l}(\mathcal{C})$ is a graded category. □

We now refine the definition of the Khovanov bracket, and the invariance theorem, to accommodate gradings.

Definition 3.1.20 In the definition of the Khovanov bracket, if T has n_+ positive crossings and n_- negative crossings, $[\![T]\!]$ has the form

$$\bullet \to [\![T]\!]^{-n_-} \longrightarrow \cdots \longrightarrow [\![T]\!]^{n_+} \to \bullet$$

where $\bullet$ is the zero object. Now add gradings to the objects of $[\![T]\!]$ by endowing $[\![T]\!]^r$ with grading $[\![T]\!]^r\{r + n_+ - n_-\}$, so that the complex is now

$$\bullet \to [\![T]\!]^{-n_-}\{n_+ - 2n_-\} \longrightarrow \cdots \longrightarrow [\![T]\!]^{n_+}\{2n_+ - n_-\} \to \bullet.$$

In the sequel the Khovanov bracket, unless otherwise stated, will be assumed to have graded objects as just described.

This allows us to recast the results as follows.

Theorem 3.1.21 *With the Khovanov bracket graded as per Definition 3.1.20, the following properties hold.*

1. *All differentials in $[\![T]\!]$ are of degree zero.*
2. *Up to degree-0 homotopy equivalence, $[\![T]\!]$ is an invariant of the tangle T. That is, if T_1 and T_2 differ by some Reidemeister moves, then there is a degree-0 homotopy equivalence $F : [\![T_1]\!] \to [\![T_2]\!]$.*
3. *The Khovanov bracket descends to a planar algebra morphism $(\mathcal{T}(s)) \to (\mathrm{Kob}(s))$ and all the planar algebra operations are of degree 0.*

Proof: We omit the proof, though each is not difficult, and more or less follow from the definitions and constructions presented above. $\square$

The construction just presented extends Khovanov's categorification of the Jones polynomial to tangles, and does so in such a way that we can work with a topological complex instead of an algebraic one. One of the benefits of working in this setting is the excellent composition properties of the Khovanov bracket. But we haven't addressed the issue of whether the bracket is a useful invariant. That is, how good is it at distinguishing tangles?

If we wished to answer this question we would need to be able to determine when two complexes in Kob are not homotopy equivalent. Showing this directly is very difficult, so we instead construct a functor to take the topological complex into an algebraic one. In the process much information about the complex is lost, but at least in the algebraic world it is easier to determine if two complexes are not homotopy equivalent – one takes homology, and if the homology groups differ, the complexes are not homotopy equivalent.

Formally, consider a functor $\mathcal{F} : \mathcal{C}ob^3_{/l} \to \mathcal{A}$ where $\mathcal{A}$ is some abelian category. It extends easily to a functor $\mathcal{F} : \mathrm{Mat}(\mathcal{C}ob^3_{/l}) \to \mathcal{A}$, and hence to a functor $\mathcal{F} : \mathrm{Kob} \to \mathrm{Kom}(\mathcal{A})$. Just as $[\![T]\!]$ is an invariant of T up to homotopy equivalence in Kob, so too is $\mathcal{F}[\![T]\!]$ an invariant up to homotopy equivalence in $\mathcal{A}$. Hence the isomorphism class of the homology groups $H^\star(\mathcal{F}[\![T]\!])$ is an invariant of T. Of course, if $\mathcal{A}$ is graded and $\mathcal{F}$ is degree respecting, then $H^\star(\mathcal{F}[\![T]\!])$ is a graded invariant of T.

We will now construct such a functor for $\mathcal{C}ob^3_{\bullet/l}(\emptyset)$. The objects of the category $\mathcal{C}ob^3_{/l}(\emptyset)$ are just disjoint unions of circles ($\bigcirc$), while the morphisms are generated by the

the cap, cup, pair of pants and upside-down pair of pants. (This is a well-know result about $(1+1)$-dimensional TQFTs, refer to any standard text for a proof of the claim.) As such a functor $\mathcal{F} : \mathcal{C}ob^3_{/l}(\emptyset) \to \mathcal{A}$ doesn't need much data. For brevity we will denote the pair of pants and upside-down pair of pants cobordisms by $\subset\!\!\!\supset$ and $\bigcirc\!\!\!-\!\!\!\bigcirc$ respectively.

Definition 3.1.22 Let V be the free abelian group generated by $\mathbf{1}$ and X. We make V into a graded abelian group by assigning $\deg \mathbf{1} = -1$ and $\deg X = 1$.

Now define a TQFT $\mathcal{F} : \mathcal{C}ob^3 \to \mathbb{Z}\,\mathrm{Mod}$ by $\mathcal{F}(\bigcirc) = V$, $\mathcal{F}(cup) = \varepsilon : \mathbb{Z} \to V$, $\mathcal{F}(cap) = \eta : V \to \mathbb{Z}$, $\mathcal{F}(\bigcirc\!\!\!-\!\!\!\bigcirc) \to m : V \otimes V \to V$, and $\mathcal{F}(\subset\!\!\!\supset) = \Delta : V \to V \otimes V$. These maps are defined by

$$
\begin{aligned}
\mathcal{F}(cup) &= \varepsilon : 1 \mapsto \mathbf{1}, \\
\mathcal{F}(cap) &= \eta : \mathbf{1} \mapsto 0, \quad X \mapsto 1 \\
\mathcal{F}(\bigcirc\!\!\!-\!\!\!\bigcirc) &= m : \begin{cases} \mathbf{1} \otimes X \mapsto X, & \mathbf{1} \otimes \mathbf{1} \mapsto \mathbf{1}, \\ X \otimes \mathbf{1} \mapsto X, & X \otimes X \mapsto 0, \end{cases} \\
\mathcal{F}(\subset\!\!\!\supset) &= \Delta : \begin{cases} \mathbf{1} \mapsto \mathbf{1} \otimes X + X \otimes \mathbf{1}, \\ X \mapsto X \otimes X. \end{cases}
\end{aligned}
$$

Proposition 3.1.23 *The functor $\mathcal{F}$ defined above is well-defined and degree respecting. It descends to a functor $\mathcal{F} : \mathcal{C}ob^3_{/l}(\emptyset) \to \mathbb{Z}\,\mathrm{Mod}$.*

In Khovanov's original construction for links [Kho99], he essentially constructed the topological cube of Definition 3.1.4, though didn't view it as an element in Kob. Instead he immediately turned the vertices and cobordisms into modules and maps and obtained a complex. The functor used to do this was precisely the one above; as such for links we have that $H^\star(\mathcal{F}[\![L]\!]) \cong Kh(L)$, where $Kh(L)$ is Khovanov's original categorification of the Jones polynomial.

This concludes the subsection. We have seen Bar-Natan's extension of Khovanov's original homology theory to the case of tangles. We saw that one of the benefits of this topological perspective is the composition properties of the Khovanov bracket – it is a morphism of planar algebras. We also saw that in the case of links, one can recover the homology groups of Khovanov's original construction by applying a certain TQFT to the construction.

We did not sketch the proof that the Khovanov bracket is actually a link invariant. We avoided this is because Kob is a difficult category to work with, and as such the proof is not elementary. As will we see in the next section, by tweaking Kob to a different category $\mathrm{Kob}_\bullet$, a tool for simplifying complexes is introduced that makes life significantly easier. Almost all of the theory developed so far carries over, so moving from Kob to $\mathrm{Kob}_\bullet$ presents no problem. As a bonus, in $\mathrm{Kob}_\bullet$ the proof of the invariance of the Khovanov bracket is much nicer, as we will see in Section 3.3.

3.2 Bar-Natan's dotted theory

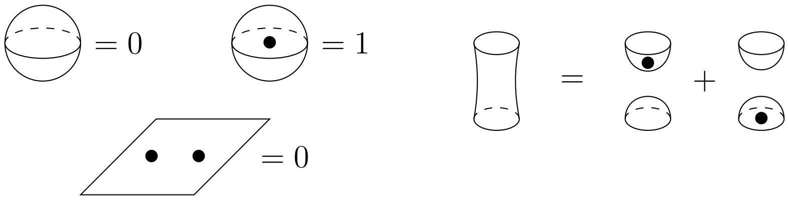
We now construct a 'dotted' version of the theory presented in the previous subsection. We change the category Kob in which the complexes lie to a dotted version $\mathrm{Kob}_\bullet$, but it is otherwise business as usual. The reason we move to the dotted setting is that it allows us to consider new homotopy equivalences. (Our main results, Theorem 4.3.1 and

Theorem 5.0.4, are built on two such isomorphisms.) In the case of links however, the dotted theory is equivalent to standard Khovanov homology.

After first constructing $\mathrm{Kob}_\bullet$, we show the equivalence of the dotted theory by constructing a TQFT (Proposition 3.2.2). We then introduce an isomorphism involving cobordisms specific to $\mathrm{Kob}_\bullet$ (Lemma 3.2.3), then close the subsection with a more general tool for simplifying complexes (Lemma 3.2.4).

To construct $\mathrm{Kob}_\bullet$, we first extend $\mathcal{C}ob^3$ to a category of dotted cobordisms $\mathcal{C}ob^3_\bullet$. This has the same objects as $\mathcal{C}ob^3$, but cobordisms are now allowed to have dots (of degree -2) on their surface. Dotted cobordisms are considered to be isotopic if their underlying cobordisms are isotopic and they contain the same number of dots on each connected component. As with $\mathcal{C}ob^3$, morphisms in $\mathcal{C}ob^3_\bullet$ are considered up to boundary-preserving isotopy.

We then quotient out $\mathcal{C}ob^3_\bullet$ by the following local relations to form $\mathcal{C}ob^3_{\bullet/l}$.



We call the last relation *neck cutting*. Note that the S, T and *4Tu* relations of $\mathcal{C}ob^3_{/l}$ hold in $\mathcal{C}ob^3_{\bullet/l}$ too: by definition S holds. When neck cutting is applied to a torus (viewed as a cobordism), it splits into a sum of dotted spheres; the sum then simplifies into a sum of two empty diagrams from which relation T follows. Similarly one applies neck cutting to either side of the $4Tu$ relation; both sides simplify to the same four-term sum of dotted disks.

Just as we denoted $\mathrm{Kom}(\mathrm{Mat}(\mathcal{C}ob^3_{/l}))$ by Kob, we will denote $\mathrm{Kom}(\mathrm{Mat}(\mathcal{C}ob^3_{\bullet/l}))$ by $\mathrm{Kob}_\bullet$.

Since the S, T and *4Tu* relations hold in $\mathcal{C}ob^3_{\bullet/l}$, the proof that $[\![T]\!]$ is an isotopy invariant of T up to degree-0 homotopy equivalences in $\mathrm{Kob}(\partial T)$ extends to $\mathrm{Kob}_\bullet(\partial T)$.

The presence of dots in the theory mean TQFTs well-defined in $\mathcal{C}ob^3_{/l}$ don't immediately extend to $\mathcal{C}ob^3_{\bullet/l}$. This said, the functor in Definition 3.1.22 which gives ordinary Khovanov homology does extend to a functor from $\mathcal{C}ob^3_{\bullet/l}$ to the category $\mathbb{Z}\,\mathrm{Mod}$ of graded $\mathbb{Z}$-modules. Instead of describing this functor directly, we will create the functor as a specialization of the following functor.

Definition 3.2.1 Let B be a finite set of points in S^1 and let $\mathcal{O}$ be an object of $\mathcal{C}ob^3_{/l}(B)$. Define a *tautological functor* $\mathcal{F}_\mathcal{O} : \mathcal{C}ob^3_{/l}(B) \to \mathbb{Z}\,\mathrm{Mod}$ on objects by $\mathcal{F}_\mathcal{O}(\mathcal{O}') := \mathrm{Mor}(\mathcal{O}, \mathcal{O}')$ and on morphisms by composition on the left. That is, send the morphism $F : \mathcal{O}' \to \mathcal{O}''$ to the map $\mathcal{F}_\mathcal{O}(F) : \mathrm{Mor}(\mathcal{O}, \mathcal{O}') \to \mathrm{Mor}(\mathcal{O}, \mathcal{O}'')$ which takes $G \in \mathrm{Mor}(\mathcal{O}, \mathcal{O}')$ to $F \circ G \in \mathrm{Mor}(\mathcal{O}, \mathcal{O}'')$.

Various specializations of this tautological functor lead to different variants of Khovanov homology. Lee's variant [Lee02], for example, is obtained by setting

$$\mathcal{F}(\mathcal{O}') := \mathbb{Z}/2\mathbb{Z} \otimes_\mathbb{Z} \mathrm{Mor}(\emptyset, \mathcal{O}')/G$$

where G is the relation setting the genus three surface to a numerical factor of 8.

Proposition 3.2.2 *Define tautological functors for objects in $\mathcal{C}ob^3_{\bullet/l}(B)$ as above. Then the functor $\mathcal{F}_\emptyset : \mathcal{C}ob^3_{\bullet/l}(\emptyset) \to \mathbb{Z}\,\mathrm{Mod}$ restricted to the subcategory $\mathcal{C}ob^3_{/l}(\emptyset)$ is the functor exhibited before in Definition 3.1.22. That is, for a link L, $H^\star(\mathcal{F}_\emptyset[\![L]\!]) \cong Kh(L)$, the original Khovanov homology of L.*

Proof: First consider an arbitrary element of $\mathrm{Mor}(\emptyset, \mathcal{O}')$ for some object $\mathcal{O}' \in \mathcal{C}ob^3_{\bullet/l}(\emptyset)$. The element can be reduced, by neck cutting, to a sum of terms in which every connected component touches at most one boundary curve. Further neck cutting can be applied so that no connected component has positive genus. Simplifying each term of the sum using the other local relations whenever possible produces a sum of cobordisms in which every connected component has exactly one boundary curve. By our simplifications these connected components are either disks or dotted disks. Hence if $\mathcal{O}'$ contains k curves, $\mathcal{F}_\emptyset(\mathcal{O}') = U^{\otimes k}$ where U is $\mathbb{Z}\{\bigcirc, \odot\}$, the free $\mathbb{Z}$-module generated by a disk and a dotted disk.

Now identify U, and V of Definition 3.1.22 via $\bigcirc = \mathbf{1}$ and $\odot = X$. To check $\mathcal{F}_\emptyset$ is the functor claimed, we must verify it takes the cup, cap, pair of pants and upside down pair of pants to the appropriate maps.

We check the map agrees on both types of pants; the other verifications are trivial.

Since the pair of pants $\bigcirc\!\!-\!\!\bigcirc$ takes two circles to a circle, the map $\mathcal{F}_\emptyset(\bigcirc\!\!-\!\!\bigcirc) : \mathrm{Mor}(\emptyset, \bigcirc\bigcirc) \to \mathrm{Mor}(\emptyset, \bigcirc)$ is defined by where it sends the generators of $U^{\otimes 2}$. As these are just pairs of (possibly dotted) disks, the cobordism obtained after composition with $\bigcirc\!\!-\!\!\bigcirc$ is isotopic to a (possibly dotted) disk. Explicitly,

$$\bigcirc\,\bigcirc \mapsto \bigcirc, \quad \bigcirc\,\odot \mapsto \odot, \quad \odot\,\bigcirc \mapsto \odot, \quad \odot\,\odot \mapsto 0.$$

With the above identifications it follows that $\mathcal{F}_\emptyset(\bigcirc\!\!-\!\!\bigcirc) = m$.

For the upside down pair of pants, the map $\mathcal{F}_\emptyset(\subset\!\!\!\supset) : \mathrm{Mor}(\emptyset, \bigcirc) \to \mathrm{Mor}(\emptyset, \bigcirc\,\bigcirc)$ sends the disks $\bigcirc$ and $\odot$ to a tube and a dotted tube respectively. After neck cutting these simplify so that the map sends

$$\bigcirc \mapsto \bigcirc\,\odot + \odot\,\bigcirc, \quad \odot \mapsto \odot\,\odot.$$

This is precisely Δ. It follows that $\mathcal{F}_\emptyset$ is the functor claimed.

Like $\mathcal{F}$, the tautological functor extends to a functor $\mathcal{F}_\emptyset : \mathrm{Kob}_\bullet(\emptyset) \to \mathrm{Kom}(\mathbb{Z}\,\mathrm{Mod})$. Since the construction of $[\![L]\!]$ in $\mathrm{Kob}_\bullet(\emptyset)$ is the same as that in $\mathrm{Kob}(\emptyset)$ (and does not involve the creation of dotted cobordisms), $H^\star(\mathcal{F}_\emptyset[\![L]\!]) \cong H^\star(\mathcal{F}[\![L]\!])$. Since $H^\star(\mathcal{F}[\![L]\!]) \cong Kh(L)$, the proposition follows. $\square$

The benefit of working in with this 'dotted' theory is that we can now consider homotopy equivalences involving dotted cobordisms, allowing us to simplify $[\![T]\!]$ in ways not previously possible. In the case of links, with one such isomorphism it is possible to simplify $[\![L]\!]$ so that all of the objects of the complex are direct sums of empty diagrams, the matrices between them taking only integer entries. The isomorphism is described in Lemma 3.2.3 below.

Although this 'delooping' process actually increases the number of subobjects in the complex, in practice many of the maps between the subobjects become isomorphisms. One can then apply a categorical version of Gaussian elimination to discard pairs of subobjects related via an isomorphism. Both tools together dramatically simplify complexes.

Perhaps unsurprisingly, these tools can be implemented on a computer. When combined with the composition properties of the Khovanov bracket, a method for rapidly

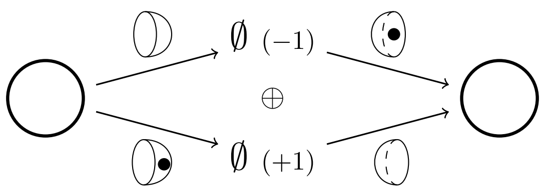
computing the Khovanov complex associated to a tangle is achieved. Indeed, as Bar-Natan found in [Bar06], an implementation of the tools, once optimized, was able to compute the Khovanov homology of the $(8,7)$ torus knot $T_{8,7}$ in a matter of minutes. (Note that this knot has 48 crossings – if the complex was computed using the original definition of Khovanov homology, the complex would contain $2.8 \cdot 10^{14}$ smoothings and $3.8 \cdot 10^{15}$ cobordisms. Obviously computing the homology of such a complex is not feasible.)

These tools are useful theoretically too – they are bread and butter for our results in the last sections. The proofs of many results in the dotted setting can be simplified with these tools, such as the invariance of the Khovanov bracket under the Reidemeister moves.

Without further ado, let us see the tools which are used constantly throughout the remainder of the paper.

Lemma 3.2.3 *If an object S in $\mathcal{C}ob^3_{\bullet/l}$ contains a closed loop l, it is isomorphic in* $\mathrm{Mat}(\mathcal{C}ob^3_{\bullet/l})$ *to the direct sum of two copies $S'\{-1\}$ and $S'\{+1\}$ of S in which l is removed, one with a degree shift of -1 and the other with a degree shift of $+1$. Symbolically,* $\bigcirc \cong \emptyset\{-1\} \oplus \emptyset\{+1\}$.

Proof: The isomorphisms are as follows.



That the composition taking $\bigcirc$ to $\bigcirc$ is the identity follows from neck cutting. The other composition is a matrix whose entries are spheres with various numbers of dots; the other local relations show this matrix is the identity. □

Lemma 3.2.4 *If $\phi : b_1 \to b_2$ is an isomorphism in some additive category $\mathcal{C}$, then the four term complex in* $\mathrm{Mat}(\mathcal{C})$

$$\cdots \longrightarrow [C] \xrightarrow{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}} \begin{bmatrix} b_1 \\ D \end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & \delta \\ \gamma & \varepsilon\end{pmatrix}} \begin{bmatrix} b_2 \\ E \end{bmatrix} \xrightarrow{\begin{pmatrix}\mu & \nu\end{pmatrix}} [F] \longrightarrow \cdots$$

is isomorphic to the complex

$$\cdots \longrightarrow [C] \xrightarrow{\begin{pmatrix}0\\ \beta\end{pmatrix}} \begin{bmatrix} b_1 \\ D \end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & 0 \\ 0 & \varepsilon - \gamma\phi^{-1}\delta\end{pmatrix}} \begin{bmatrix} b_2 \\ E \end{bmatrix} \xrightarrow{\begin{pmatrix}0 & \nu\end{pmatrix}} [F] \longrightarrow \cdots.$$

Both are homotopy equivalent to the complex

$$\cdots \longrightarrow [C] \xrightarrow{(\beta)} [D] \xrightarrow{(\varepsilon - \gamma\phi^{-1}\delta)} [E] \xrightarrow{(\nu)} [F] \longrightarrow \cdots.$$

Here C, D, F and F are arbitrary columns of objects in $\mathcal{C}$ and all Greek letters (other than ϕ) represent arbitrary matrices of morphisms in $\mathcal{C}$ (having the appropriate dimensions, domains and ranges); all matrices appearing in these complexes are block-matrices with blocks as specified. The claim still holds if b_1 and b_2 are direct sums of objects, provided the matrix ϕ is invertible.

Proof: The 2×2 (block) matrices in the first two complexes above differ by invertible row and column operations – that is, by a change of basis. When the corresponding operations are performed on $\begin{pmatrix}\mu & \nu\end{pmatrix}$ and $\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$ these change to $\begin{pmatrix}\mu - \nu\gamma\phi^{-1} & \nu\end{pmatrix}$ and $\begin{pmatrix}\alpha - \phi^{-1}\delta\beta \\ \beta\end{pmatrix}$ respectively. Since the differentials square to 0 we have $\mu\phi - \nu\gamma = 0 = \phi\alpha - \delta\beta$, from which the isomorphism follows.

Note that the second complex segment is the direct sum of the third complex segment and

$$\cdots \longrightarrow 0 \longrightarrow [b_1] \xrightarrow{\quad (\phi) \quad} [b_2] \longrightarrow 0 \longrightarrow \cdots.$$

Since ϕ is an isomorphism this complex is contractible. (That is, homotopy equivalent to the zero complex.) It follows that the second (and hence the first) complex segment is homotopy equivalent to the third complex segment. $\square$

At first glance the tools may seem humble, but in reality they provide a way to radically simplify complexes that are otherwise inaccessible. Examples of delooping and Gaussian elimination in action may be found in the next subsection (and subsequent section).

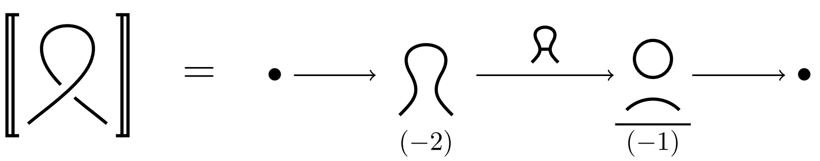
3.3 The invariance of the Khovanov bracket under the Reidemeister moves

We now prove the invariance of the Khovanov bracket in $\mathrm{Kob}_\bullet$ under the Reidemeister moves. There are three separate arguments we need to make, but our approach in each case is the same. We compute the Khovanov complexes associated to both sides of a Reidemeister move, then simplify them using the tools exhibited in the previous section. Since delooping and Gaussian elimination are both homotopy equivalences, if two complexes simplify down to the same complex, the complexes themselves are homotopy equivalent.

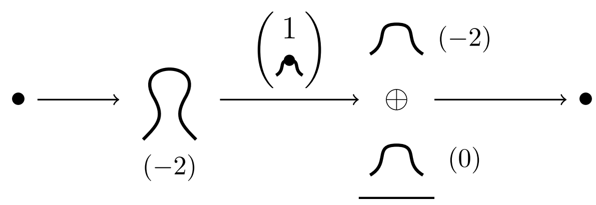
This is in contrast to the corresponding proofs in Kob [Bar04], where explicit homotopy equivalences are constructed to prove R1 and R2, while R3 is proved using the map in R2 and various cone constructions.

The length of the arguments presented here increases with the number of crossings in the diagrams; as such we work through the Reidemeister moves in order.

Proof of R1 invariance: The Khovanov complex of the 'twisted' side of R1 is as follows.

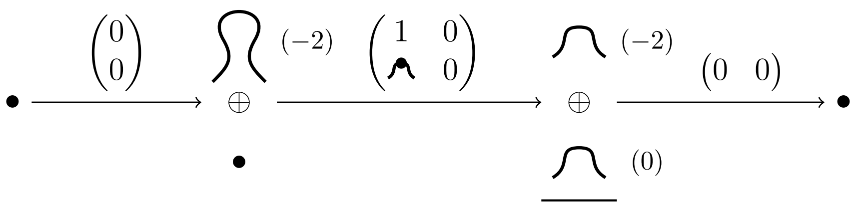


We deloop the last term to obtain the following isomorphic complex.

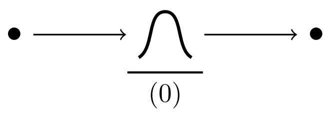


Note that when we composed the saddle cobordism with the various cups of the delooping isomorphism, the shape of the cobordism we obtain resembles an arm of a cactus. As such we can isotope this arm back into the main 'curtain' of the cobordism, obtaining the cobordisms shown.

The identity cobordism is an isomorphism, so we can further simply the complex with Gaussian elimination. To eliminate any ambiguity about how Lemma 3.2.4 may be applied, let us write the complex in the form given in the lemma.



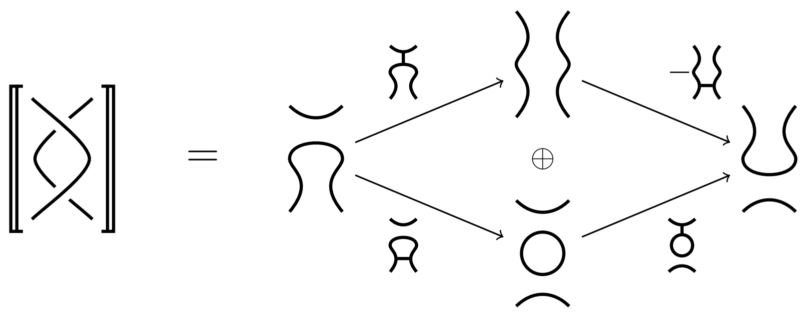
When the lemma is applied, we obtain



.

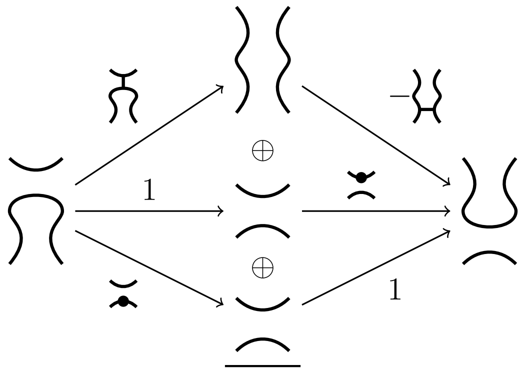
This is precisely the Khovanov complex of the crossingless tangle corresponding to the other side of R1. □

Proof of R2 invariance: The Khovanov complex of the R2 diagram with crossings is as follows.

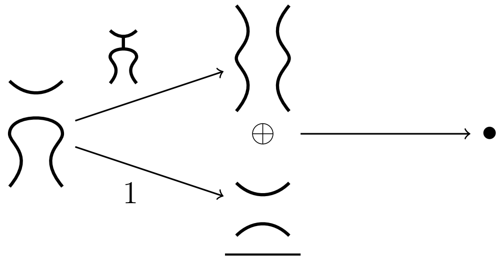


To avoid clutter we've dropped the internal gradings and haven't included the zero objects

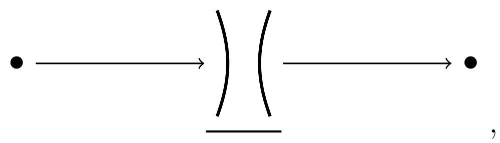
on the ends. Delooping the only term we can gives us the following complex.



One can instantly see from the isomorphisms here that all but one of the terms will cancel. Applying Gaussian elimination to the isomorphism on the right of the complex gives



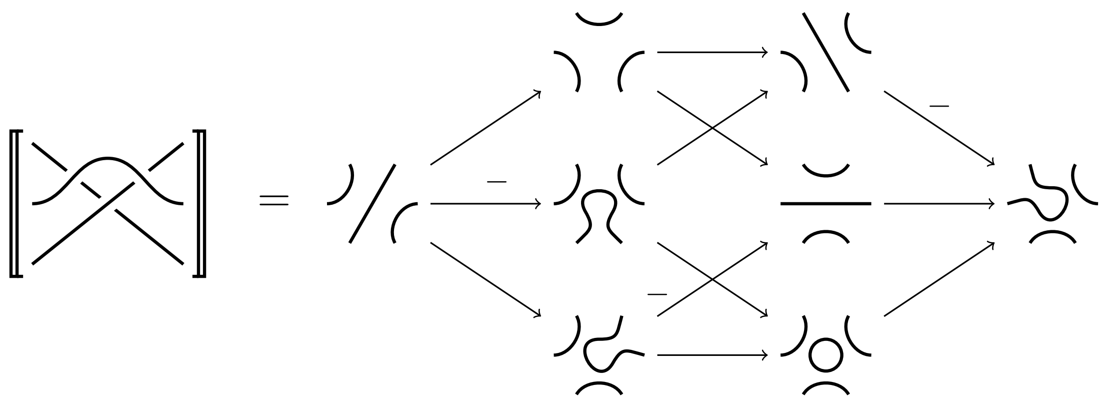
By eliminating the remaining isomorphism we obtain

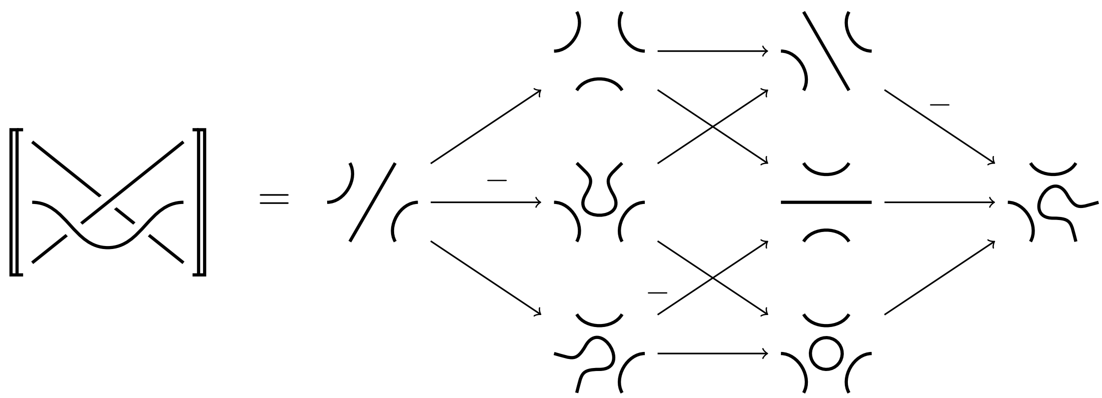


the complex associated to the 'uncrossed' R2 diagram. □

Proof of R3 invariance: Both sides of the R3 relation contain three crossings and as such the Khovanov complexes are larger than those previously exhibited. The proof is not too bad though.

The unreduced Khovanov complexes of both sides of R3 are below.

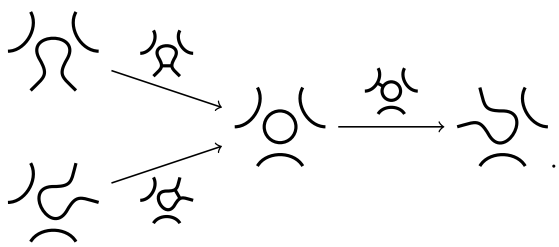




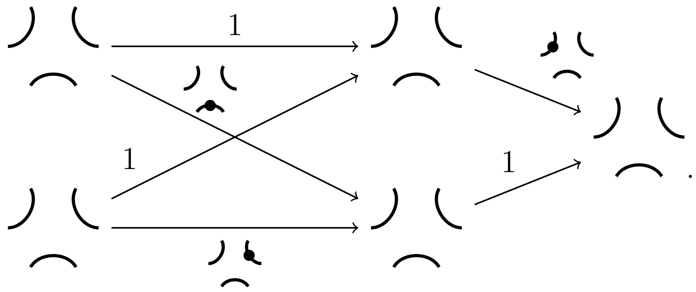
For clarity we've dropped the cobordisms between the terms (each is a saddle) and we've taken the liberty to isotope several of the objects. The minus signs indicate saddles with a negative coefficient. We've also dropped the quantum gradings and homological height information – these depend on the orientation of the components but do not change the argument.

Notice the similarity in the arrangement and type of the subobjects in the complexes.

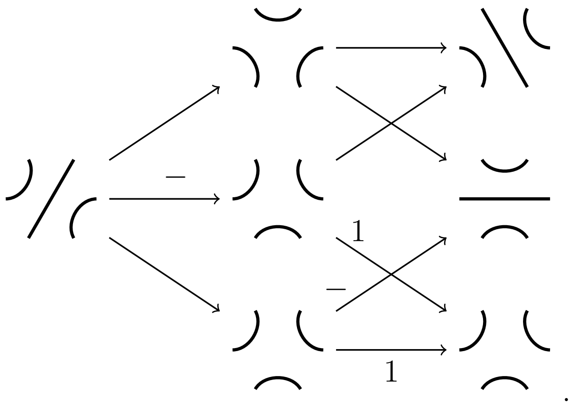
Let us first simplify the top complex. There is only one indecomposable object in the complex with a loop. The maps going into and out of this object are



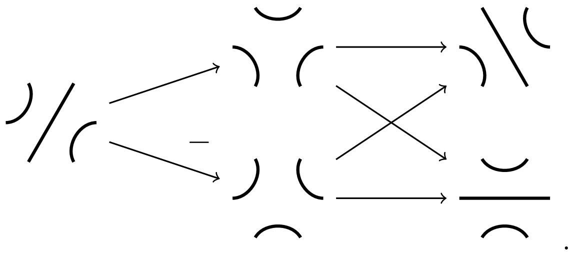
Upon delooping, three isomorphisms are introduced into the complex. The previous part of the complex becomes



Eliminating the right-most isomorphism reduces the whole complex to

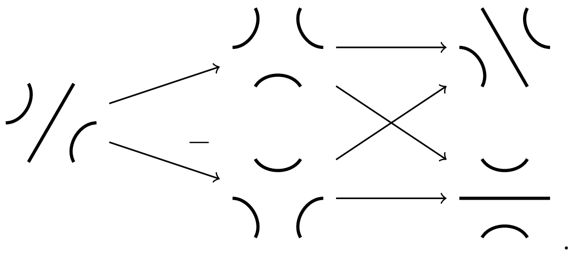


(As before, the unmarked maps are saddles.) After eliminating the bottom isomorphism the complex simplifies to



Note that the zero map between the middle diagrams in the second and third columns of the complex has changed to a saddle.

We now examine the Khovanov complex associated to the other side of R3. In exactly the same way as the previous case, this simplifies to



The only difference between this and the other simplified diagram is the location of the negative sign. The following lemma shows the two are the same. □

Lemma 3.3.1 *Let $(C_\bullet, d_\bullet)$ be a chain complex, and let $(C_\bullet, d'_\bullet)$ denote a chain complex obtained from $(C_\bullet, d_\bullet)$ by negating one of the differential maps. Then $(C_\bullet, d_\bullet) \cong (C_\bullet, d'_\bullet)$.*

Proof: This is trivial, consider the following chain map.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "1"] & C_n \arrow[r, "d_n"] \arrow[d, "1"] & C_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "-1"] & C_{n-2} \arrow[r, "d_{n-2}"] \arrow[d, "-1"] & C_{n-3} \arrow[r] \arrow[d, "-1"] & \cdots \\
\cdots \arrow[r] & C'_{n+1} \arrow[r, "d_{n+1}"'] & C'_n \arrow[r, "-d_n"'] & C'_{n-1} \arrow[r, "d_{n-1}"'] & C'_{n-2} \arrow[r, "d_{n-2}"'] & C'_{n-3} \arrow[r] & \cdots
\end{tikzcd}$$

□

4 The Khovanov homology of rational tangles

In this section we prove Theorem 4.3.1, which together with Theorem 5.0.4 essentially provides a complete description of the Khovanov homology of rational tangles. The theorem in this section describes how the homotopy class of the Khovanov complex of a rational tangle has a particularly simple, and canonical, representative. We describe this representative very explicitly, and as we find in Section 5, the underlying subobjects of the representative can surprisingly be described via the reduced Burau representation of B_3.

In Section 4.1 we use the tools in Section 3.2 to compute the Khovanov complex of a integer tangle (Proposition 4.1.1). The proof is by induction, and we find that the complexes $[\![n]\!]$ all have a simple, and periodic, representative. We then introduce a notation with which one can easily describe and visualize the Khovanov complex of a rational tangle. We illustrate the notation and the result of the section by computing $Kh(7_1)$ by hand (Example 4.1.3).

In Section 4.2 we describe a 'square isomorphism' (Proposition 4.2.1), and how it can simplify previously inscrutable complexes into the aforementioned canonical form.

Finally, in Section 4.3 we combine these results to obtain Theorem 4.3.1. We illustrate the theorem by easily computing $Kh(8_2)$ by hand in Example 4.3.3.

In Section 2 we saw that every positive (negative) rational tangle is isotopic to a standard form $\langle a_1, a_2, \ldots, a_n\rangle$ where all a_i are non-negative (non-positive) (Corollary 2.2.9). In such a form a positive (negative) rational tangle can be constructed from a finite sequence of additions and products with the $[+1]$ ($[-1]$) tangle. This means that the Khovanov complex of a rational tangle can be constructed inductively from $[\![\pm 1]\!]$ by a sequence of intermediate complexes, where each is obtained from the next by adding a crossing then immediately simplifying the resulting complex. Theorem 4.3.1 describes precisely how to obtain $[\![T + [1]]\!]$ or $[\![T * [1]]\!]$ from $[\![T]\!]$ by breaking apart the 'morphism string' (Section 4.3) of $[\![T]\!]$ into several subwords, applying certain rules to the subwords, and concatenating the result.

For the rest of the paper, we'll work with positive tangles. The results for the case of negative tangles are completely analogous. We will also use the notation $[\![a_1, a_2, \ldots, a_n]\!]$ to denote the Khovanov complex of the rational tangle $\langle a_1, \ldots, a_n\rangle$.

For those hard-pressed for time, the section can be summarized in a triptych of diagrams: Figure 4.1.1, Figure 4.2.5 and Figure 4.3.1. If you understand these, you've understood this section.

4.1 The Khovanov complex of integer tangles

In this section we compute the Khovanov complex $[\![n]\!]$ of the integer tangle $[n]$. We do this inductively, using only the tools of delooping and Gaussian elimination described in the previous section. The process is summarized in the illustration over the page.

Integer tangles are the building blocks of rational tangles, and understanding their Khovanov homology will allow us to understand the Khovanov homology of rational tangles in general. In particular, the simplification of the Khovanov complexes of integer tangles, described in Proposition 4.1.1 below, is one of the main ingredients in Theorem 4.3.1, the main result of this section.

In this section, and in later parts, we will often use Lemma 3.3.1 to simplify results.

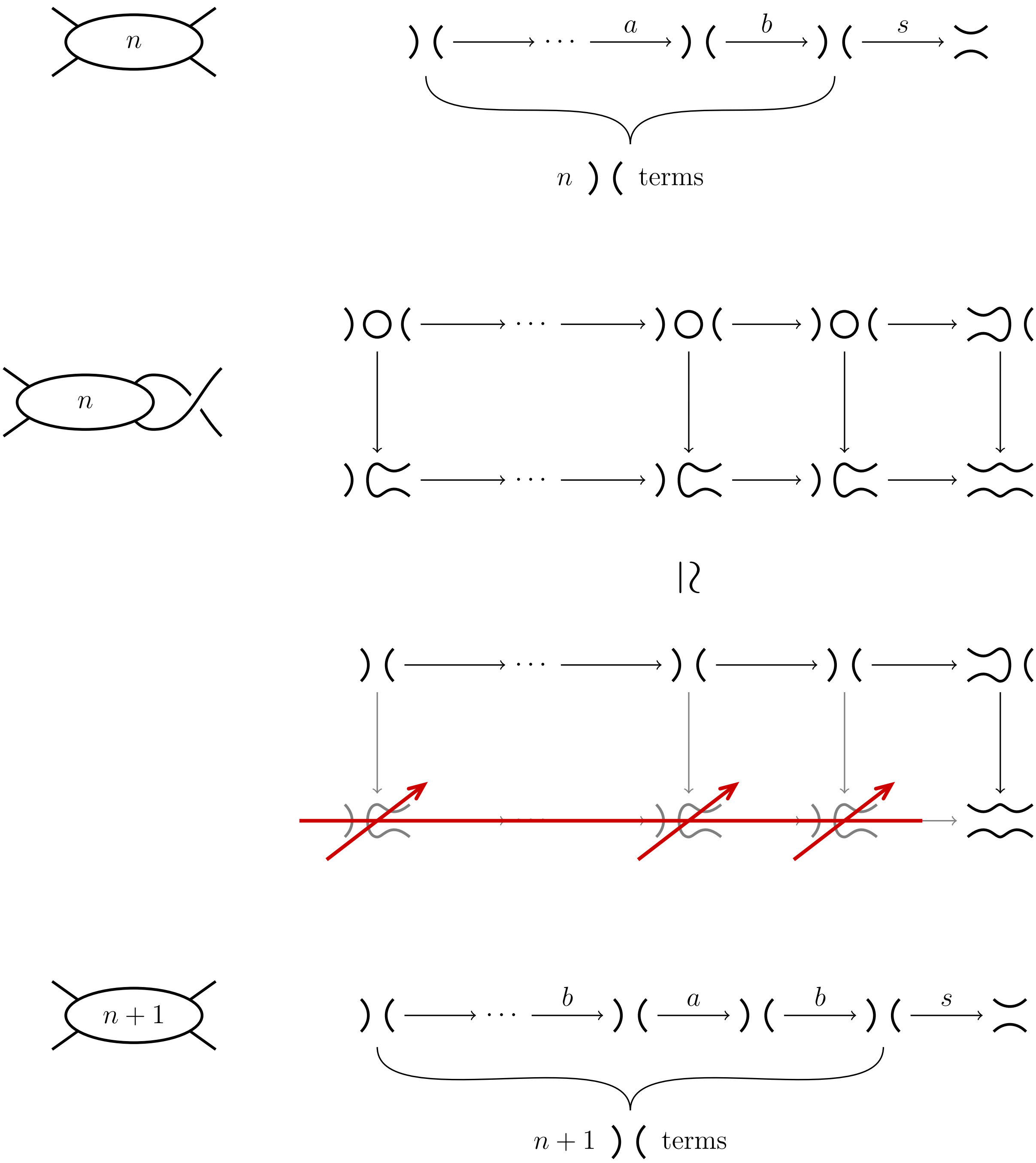
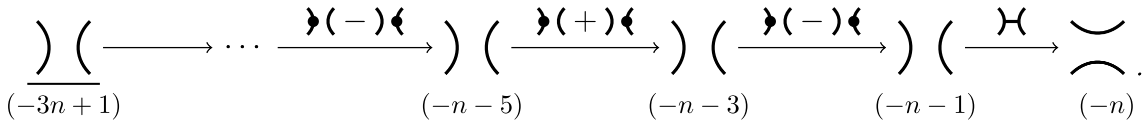


Figure 4.1.1: When an integer tangle is extended by another crossing, the Khovanov complex picks up another term. This small change in the size of the complex is due to the cancellation of nearly an entire row.

Proposition 4.1.1 *Let $n > 0$ and $[n]$ have orientation type II (subsection 2.3). Then $[\![n]\!]$ is homotopy equivalent to*



Proof: The case $n = 1$ follows directly from the definition of Khovanov homology. The case $n = 2$ is similar to the proof of the invariance of the Khovanov bracket under R2. Namely, we write $[2] = [1] + [1]$ and construct the planar arc diagram D corresponding to tangle addition (Figure 4.1.2). Since the Khovanov bracket is a planar algebra morphism, $[\![2]\!] = [\![D([1],[1])]\!] = D([\![1]\!],[\![1]\!])$. The complex $D([\![1]\!],[\![1]\!])$ is constructed in Figure 4.1.2 below. We simplify the complex as follows. Delooping the object in the NW corner gives

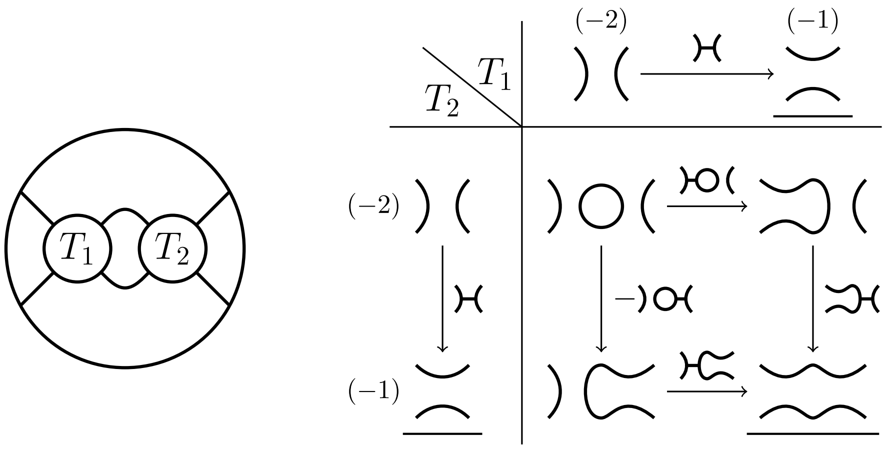
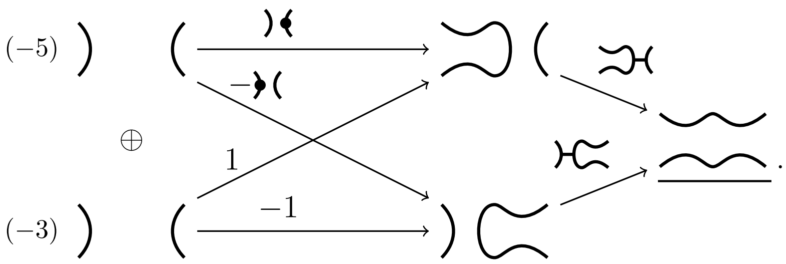
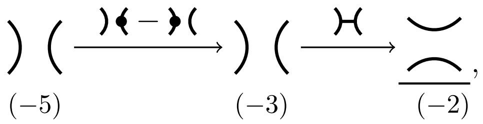


Figure 4.1.2: LEFT The integer tangle [2] is the result of placing the [1] tangle in both holes of the planar arc diagram illustrated. RIGHT Calculating $[\![2]\!]$ from $[\![1]\!]$ and the planar arc diagram to the left.

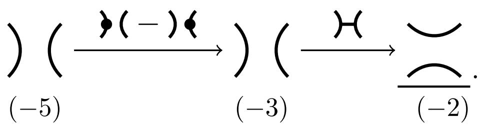
us



After Gaussian elimination, this simplifies to



which is isomorphic to



Now assume the claim is true for some $n \geq 2$. By using the same 'addition' planar arc diagram in Figure 4.1.2 above with $T_1 = [n]$ and $T_2 = [1]$, we have $[\![n+1]\!] = D([\![n]\!], [\![1]\!])$. This complex is as follows.

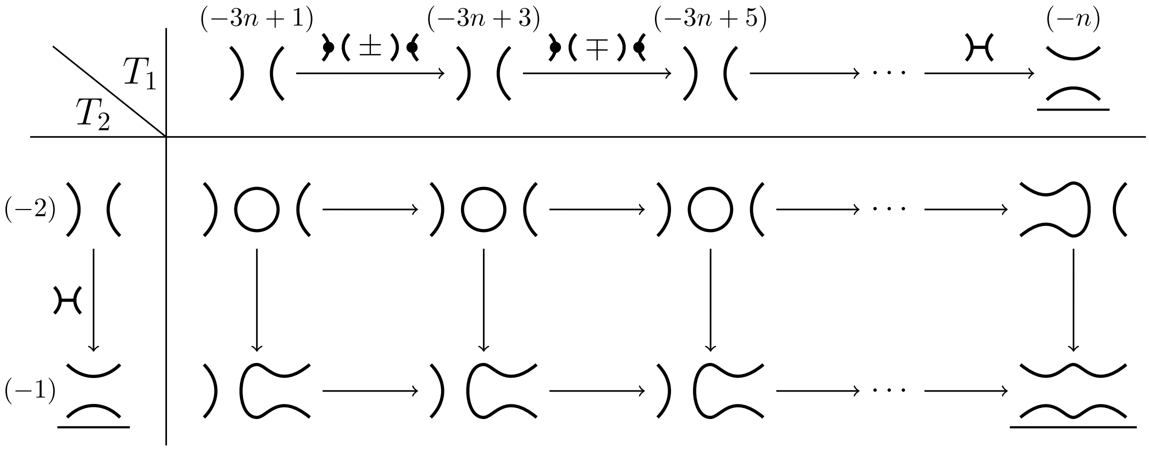
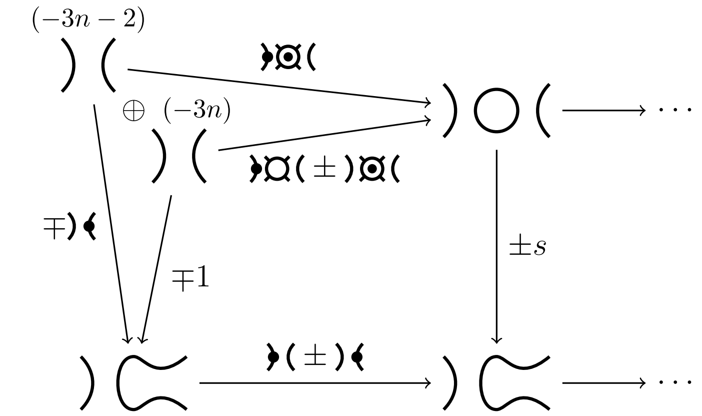
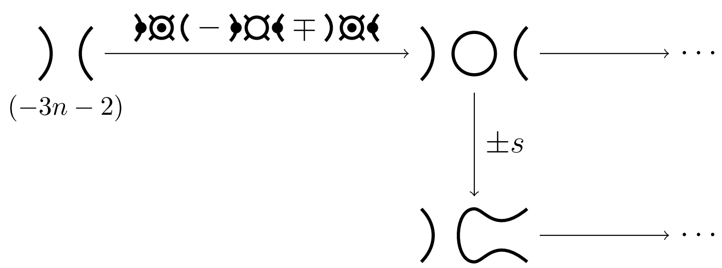


Figure 4.1.3: The complex $[\![n+1]\!]$ can be computed from $[\![n]\!]$ and $[\![1]\!]$ using the same planar arc diagram in Figure 4.1.2. For readability the morphisms of the complex have been omitted.

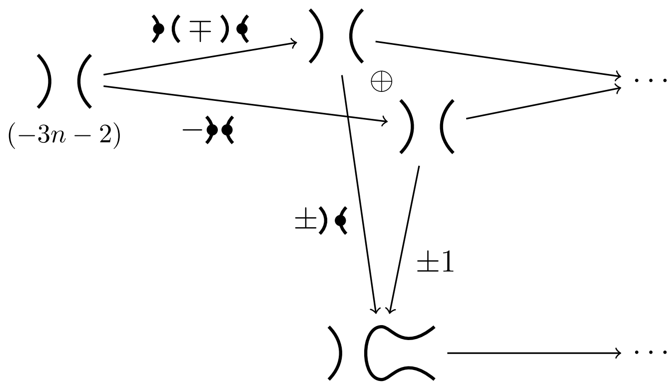
We simplify the NW corner of the complex by delooping, and then applying Gaussian elimination. The complex obtained after delooping is as follows. (To avoid clutter, in these complexes, and in the sequel, we often denote a saddle morphism by s.)



We then apply Gaussian elimination and obtain the following complex.



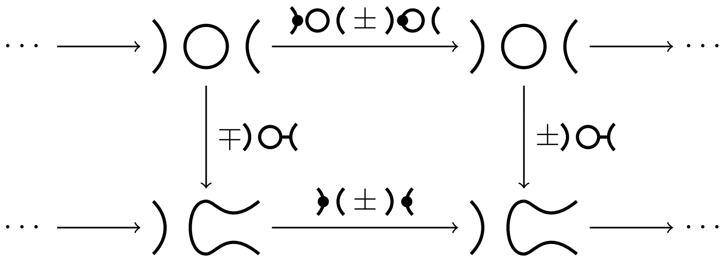
Note that the grading on the subobject in the NW corner has decreased by 1 as a result of the simplification. The morphism out of this subobject may appear complicated, but simplifies when the next term is delooped.



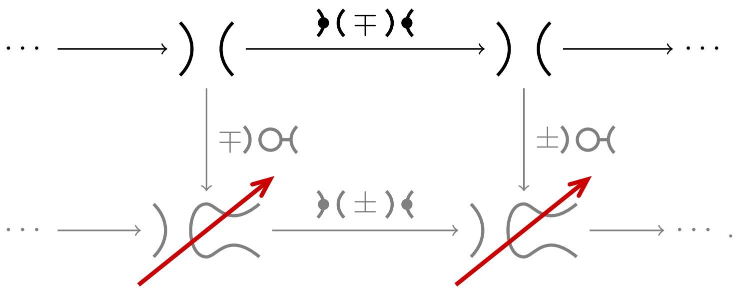
A further application of Gaussian elimination clearly removes two more subobjects from this complex.

So far, the complex in Figure 4.1.3 has simplified as follows. Of the four west-most subobjects, the two northern subobjects have been delooped, their quantum grading decreasing by 1, and the two southern subobjects have been eliminated. After each northern subobject was delooped, an isomorphism was introduced, which allowed us to eliminate the subobject directly south of it.

We continue to move left to right through the complex using this method – delooping each successive tangle in the north row and eliminating the object south of it. In general, when part of the complex has the form



it simplifies to



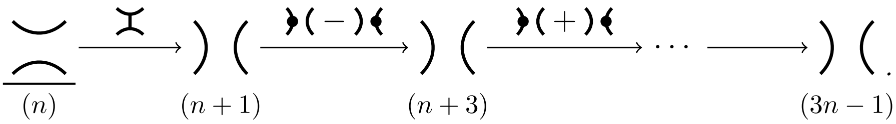
That is, we eliminate the southern objects, deloop the northern objects, and 'conjugate' the maps between the northern objects. By this, we mean the maps $D(\rangle(\pm)\langle, 1)$ between the northern objects simplify to $\rangle(\mp)\langle$.

After working down the chain complex, eventually only two subobjects on the southern row remain. These subobjects, together with those directly north of it, form a square consisting only of saddle maps. This square is the complex associated to $[\![2]\!]$, consisting only of saddle maps. The rest of the complex, consisting of the terms we have delooped

is attached to this square like a tail. The square is simplified in exactly the same way as we did previously for the [2] tangle, from which the proposition follows. □

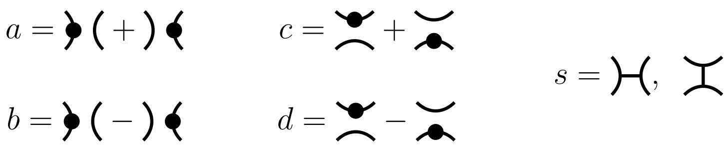
By essentially the same proof, we have the corresponding result for negative integer tangles.

Proposition 4.1.2 *Let $n < 0$ and $[n]$ have orientation type II. Then $[\![n]\!]$ is homotopy equivalent to*



We have just seen that the Khovanov complex of an integer tangle can be dramatically simplified: eventually all the objects of the complex are indecomposable, and there are at most three types of non-zero morphisms between these.

Rational tangles in general do not have complexes as elementary as this. However, it is true that any non-zero map between two indecomposable subobjects in the complex of a rational tangle is (up to sign) one of the six morphisms below.



This property is a consequence of Theorem 4.3.1, and we will often use these abbreviations in the sequel to minimize clutter when depicting complexes.

We will also not worry about the signs of morphisms between indecomposable subobjects in general; this information can be varied by Lemma 3.3.1 and is not needed to prove Theorem 5.0.4.

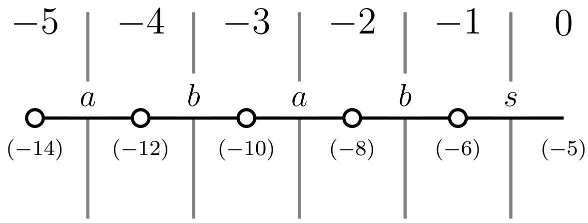
The fact that there are only a small number of morphisms between subobjects mean the complexes themselves can be represented in a more efficient way.

To each Khovanov complex of a rational tangle we can construct a *dot diagram*. These contain all the information of the complex but in an easier-to-read form. This notation is useful to visualize the Khovanov complex associated to a rational tangle, and simplifies calculations. By using it we can illustrate how the complex associated to a rational tangle changes when crossings are added to the tangle, as we do in Figure 4.3.1. The chain complexes in this figure contain hundreds of subobjects – it would be impractical to write these out as a sequence of objects and morphisms in the traditional way.

To construct a dot diagram for a given complex, begin by drawing a number of vertical lines; the spaces between them represent particular homological heights. Denote each $[\infty]$ and $[0]$ subobject in the complex by a circle or dot respectively in the appropriate homological height. If there is a non-zero morphism between two indecomposable objects in the complex, connect the corresponding objects by a line in the dot diagram.

Additional data can be added to the diagram to describe the homological heights, gradings, and morphisms. A dot diagram of $[\![5]\!]$ with such information is illustrated

below.

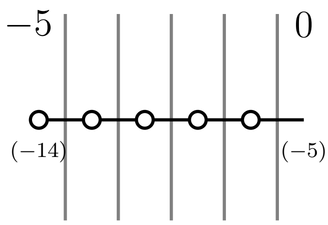


Here the integers at the top denote the various homological degrees, the integers in the brackets represent the quantum gradings of the subobjects, and the letters represent the morphisms using the abbreviations above.

All we've done so far is repeat the information contained in the complex corresponding to [5], but much of this is redundant. By removing all but the essential information we can draw a simpler dot diagram.

- If we assume crossing a vertical line from left to right represents a positive change by 1 in homological degree, we only require one integer at the top of a dot diagram to calibrate the other homological degrees.
- The morphisms a, b, c, d take objects with quantum grading n to $n+2$, and saddles s take objects with quantum grading n to $n+1$. As these are the only morphisms present between subobjects (up to sign) in the complex of a rational tangle (proved later), provided the dot diagram is connected (in the graph-theoretical sense), the grading of one subobject determines the others.
- Not all morphisms need to be described. Since $\partial^2 = 0$, the identity of many morphisms can be determined from others.

The last point is best illustrated with an example. Let us redraw the dot diagram of $[\![5]\!]$ above.

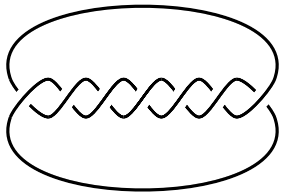


We've included more than one homological height and grading for readability, but the point here is that the morphisms don't need to be labeled, since their identity can be determined.

Explicitly, the morphism between homological heights −1 and 0 can only be a saddle, for of the six possible non-zero morphisms between subobjects previously exhibited, no other type of morphism between a $[\infty]$ tangle and a $[0]$ tangle exists. The morphism between homological heights −2 and −1 must then be b: a map between $[0]$ tangles is either a or b but cannot be the former since we require $\partial^2 = 0$. Similarly, there are two choices for the morphism between heights −3 and −2. As $b \circ a = 0 \neq b \circ b$, it follows that the morphism must be a. The other morphisms can be determined from similar logic.

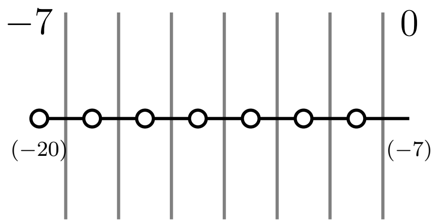
We now conclude the section with a calculation.

Example 4.1.3 Let's compute $Kh(7_1)$, the Khovanov homology of the 7_1 knot pictured below.



The knot 7_1 is the numerator closure of the tangle [7]. An orientation of 7_1 induces an orientation on the tangle, making it of type II.

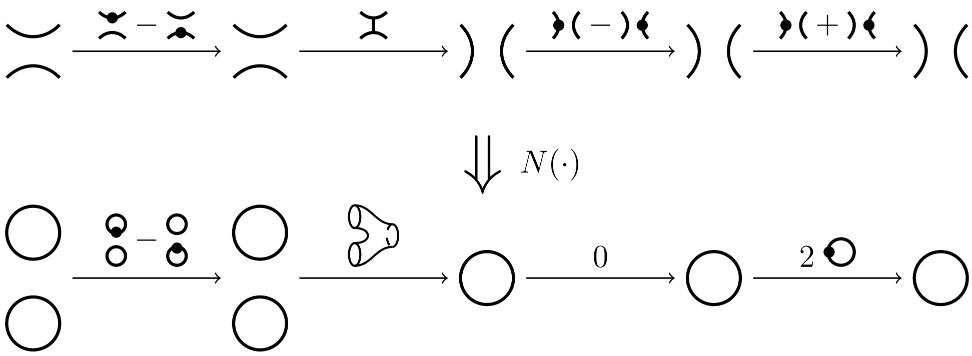
As per Proposition 4.1.1, the Khovanov complex of this tangle is the following.



We can recover $[\![7_1]\!]$ by placing $[\![7]\!]$ in the 'figure-8' planar arc diagram illustrated below in Figure 4.1.4. When we do this $[\infty]$ subobjects become circles while [0] subobjects become pairs of circles. Similarly morphisms change, as illustrated below.

Note that b morphisms become zero morphisms: recall that b consists of the formal difference of two cobordisms, each of which consists of two sheets in which one sheet contains a dot. When either of these is placed in the figure-8 tangle, the ends of the sheets are joined, resulting in a cylinder with a dot. Although the location of the dots on the cylinders are different, in $\mathcal{C}ob^3_{\bullet/l}$ the two morphisms are considered identical, so their difference is zero.

Since placing a tangle in the figure-8 planar arc diagram is the same as taking its numerator closure, we'll abuse notation and denote by $N(C)$ the result of putting a complex C in this planar arc diagram.



We can represent complexes $N([\![T]\!])$ by a dot diagram too; in an abuse of notation we'll use the same symbols as before, except now circles represent circles and dots represent two circles. The resulting dot diagram for $[\![7_1]\!]$ is in Figure 4.1.4 below. (We obtained this by simply removing any edge from the dot diagram of $[\![7]\!]$ that was a b map.)

The complex $[\![7_1]\!]$ thus simplifies into a direct sum of four components, three of which are the same (ignoring homological and quantum grading shifts).

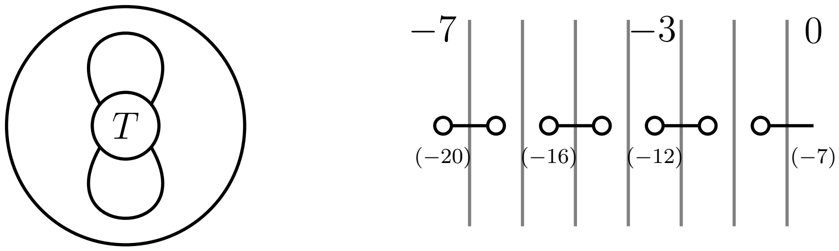
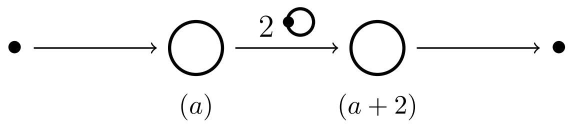


Figure 4.1.4: When numerator closure $N([\![T]\!])$ of the Khovanov complex $[\![T]\!]$ of a rational tangle T is taken (placed in the figure-8 planar arc diagram illustrated), the complex splits into a direct sum of several types of components, as shown for $N([\![7]\!])$.

Let us calculate the homology of the complex corresponding to the ○—○ component of the dot diagram. That is, the homology of the following complex.



(As before, the numbers in the brackets are the quantum gradings; the dots at the beginning and end represent the zero object in $\mathrm{Mat}(\mathcal{C}ob^3_{\bullet/l}(4))$.) We can deloop both ends to simplify it even further. The complex obtained is

$$\bullet \longrightarrow \begin{bmatrix} \emptyset\ (a-1) \\ \emptyset\ (a+1) \end{bmatrix} \xrightarrow{\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}} \begin{bmatrix} \emptyset\ (a+1) \\ \emptyset\ (a+3) \end{bmatrix} \longrightarrow \bullet.$$

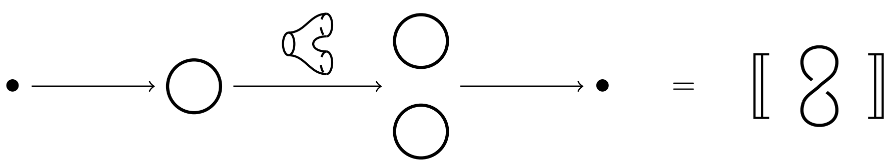
Before we take homology, we need to apply a TQFT. Under the tautological functor defined in Proposition 3.2.2, the complex becomes

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}} \mathbb{Z} \oplus \mathbb{Z} \to 0.$$

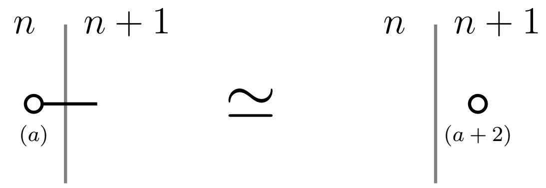
The homology groups of this complex are trivial to compute. We obtain

$$Kh^n(\circ\!\!-\!\!\circ) = \mathbb{Z}_{(a-1)}, \qquad Kh^{n+1}(\circ\!\!-\!\!\circ) = \mathbb{Z}_{(a+3)} \oplus \mathbb{Z}/2\mathbb{Z}_{(a+1)}.$$

(The subscripts here indicate the quantum grading of the group.) To complete the calculation of $Kh(7_1)$, we only need to calculate the homology of the last component in the dot diagram of $[\![7_1]\!]$. But there is no need to deloop this and do further matrix calculations. Rather, note that the complex corresponding to this is just the Khovanov complex of a diagram of the unknot with one crossing.



By keeping track of grading and homological shifts, we obtain the following homotopy equivalence.



The calculation is now easy, since the Khovanov homology of the unknot is trivial to calculate.

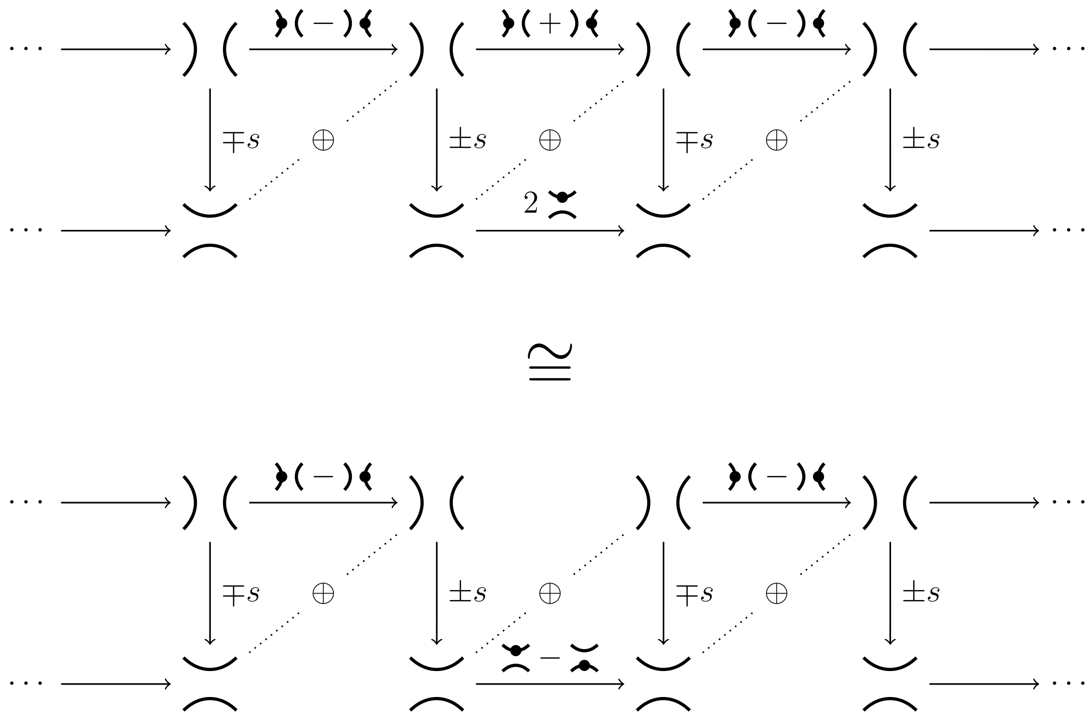
Assembling this information together and substituting the relevant grading and homological height information, we obtain

$$Kh^n(7_1) = \begin{cases} \mathbb{Z}_{(-21)} & \text{if } n = -7 \\ \mathbb{Z}/2_{(-19)} \oplus \mathbb{Z}_{(-17)} & \text{if } n = -6 \\ \mathbb{Z}_{(-17)} & \text{if } n = -5 \\ \mathbb{Z}/2_{(-15)} \oplus \mathbb{Z}_{(-13)} & \text{if } n = -4 \\ \mathbb{Z}_{(-13)} & \text{if } n = -3 \\ \mathbb{Z}/2_{(-11)} \oplus \mathbb{Z}_{(-9)} & \text{if } n = -2 \\ \mathbb{Z}_{(-7)} \oplus \mathbb{Z}_{(-5)} & \text{if } n = 0 \\ \{0\} & \text{otherwise.} \end{cases}$$

(This agrees with the values published in the literature [KA].)

4.2 A helpful isomorphism

We now come to an important isomorphism. If there is any piece of machinery to take away from the paper, this is it, since the entire paper is based on it. The isomorphism is as follows.

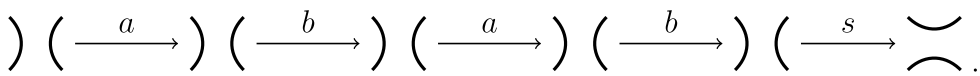


We've presented these complexes using the same format that we used in the previous section for readability. Namely, the objects of the complexes consist of the diagonal direct sums of subobjects in the diagram, and the non-zero maps between these are indicated. As with the previous section we will drop the direct sum symbols which indicate which subobjects are in the same homological height; this information can be inferred from the arrows in the diagram.

Proposition 4.2.1 *The complexes above are isomorphic in* $\mathrm{Kom}(\mathrm{Mat}(\mathcal{C}ob^3_{\bullet/l}(4)))$.

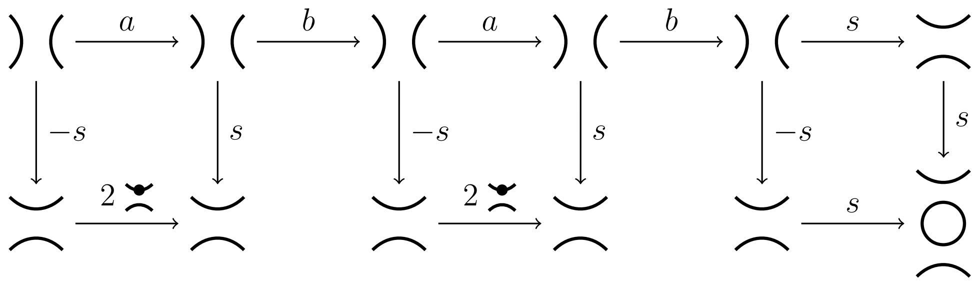
We will see the proof of this shortly, but let us first motivate why this 'square isomorphism' is glorious for our purposes.

From the previous section we know that the complexes of integer tangles $[\![n]\!]$ are essentially all the same: a chain of a and b maps followed by a saddle at the end or beginning. The complex $[\![5]\!]$ for instance, modulo homological shifts and grading information, is



If we add additional crossings to [5], how do the corresponding Khovanov complexes change? If we add [5] and [1] we obtain [6], and we know what the Khovanov complex of this is. But what if we were to add a crossing to the bottom of the [5] tangle?

Example 4.2.2 Let us calculate $[\![5, 1]\!]$ without worrying about the quantum gradings or the precise homological degrees (but we will of course keep track of the relative homological degrees). With [5] as T_1 and [1] as T_2 in the planar arc diagram corresponding to the product of two tangles (Figure 4.2.2 below), we obtain the following complex.

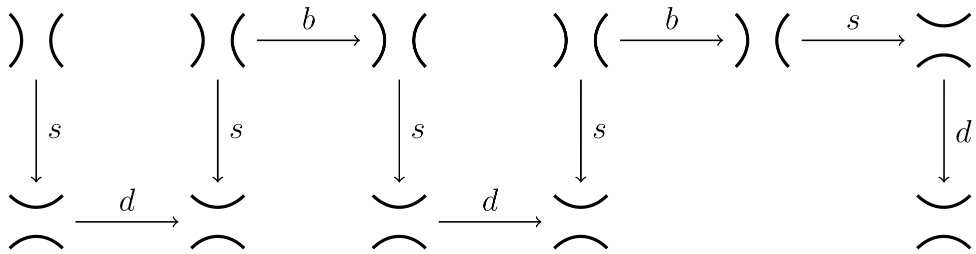


Note the zero maps in the complex. These are the maps $D(b, I) : D([\infty], [0]) \to D([\infty], [0])$. Although the cobordism b consists of the difference of two pairs of vertical sheets with dots on different sheets, in $D(b, I)$ the sheets are are connected. The cobordism $D(b, I)$ consists of the difference of two pairs of vertical sheets, both of which contain a dot on the same component, hence is zero.

The right-most anticommuting square of the complex, consisting only of saddles, simplifies in exactly the same way as in the calculation of $[\![2]\!]$ in the proof of Proposition 4.1.1. But the rest of the complex has now become complicated – at least to the extent that if we were to try adding another crossing, further calculations by hand would be messy and unreasonable. For instance, assume we added another crossing to the bottom of $\langle 5, 1\rangle$ to obtain $\langle 5, 2\rangle$. By treating $\langle 5, 1\rangle$ as T_1 and [1] as T_2, if we were to construct $[\![5, 2]\!]$ using the same 'product' planar arc diagram as before, the nodes and maps of the complex we would construct would now consist of vectors and matrices.

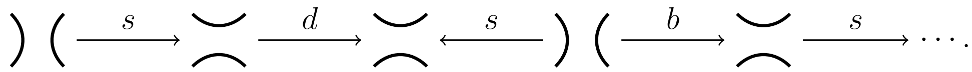
Using the square isomorphism illustrated before bypasses this problem. With it, the structure of $[\![T]\!]$ for a rational tangle T admits a far nicer description.

If we allow ourselves to use the square isomorphism, the complex $[\![5,1]\!]$ simplifies to the following representative.



(We also went a step further and removed the minus signs on the saddle morphisms.)

We have essentially made the complex 1-dimensional, a string of indecomposable subobjects connected via arrows. That is, it is a zig-zag complex (Definition 1.0.1). We can write it as



If we assume a complex has a presentation in this form – a string of objects and morphisms but with arrows going forwards *and* backwards – can we determine which objects lie in the same homological height? The answer is trivially yes: arrows in the unreduced, Khovanov complex take objects in homological height n to objects in homological height $n+1$. Delooping and Gaussian elimination rearrange and eliminate objects, but don't shift homological heights. Neither does our square isomorphism, so after simplification, every arrow still spans one homological height.

In practice, recovering the complex by hand from such a 'string presentation' of a complex is easy – simply construct the corresponding dot diagram, as illustrated below. (However in this situation, and in the sequel, we slope the lines to make sure the diagram doesn't become cluttered. When we construct a dot diagram, while reading left to right along the string presentation we construct circles and dots while gradually moving north up the diagram. A left arrow encountered in the string presentation corresponds to moving left in the dot diagram; a right arrows encountered indicates going right.) Such a dot diagram for $[\![5,1]\!]$ is illustrated below.

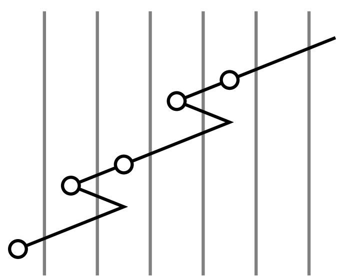


Figure 4.2.1: A dot diagram of $[\![5,1]\!]$. As with all rational tangles, it can easily be constructed from the string presentation of $[\![5,1]\!]$.

Notice that in the final dot diagram, we didn't label the morphisms. There was no need to, since all could be determined: each of the five 'straight' segments in the dot diagram contain a saddle, which determine the rest of the morphisms along the segment. (As we will soon see, there is no need to label the morphisms in the case of a general rational tangle too.) This concludes the calculation.

The reason why these string presentations of a complex are useful is that, for computations, we can treat them as we would a normal complex. An example will clarify what we mean.

Example 4.2.3 We can compute $[\![0,2,2,1]\!]$ from $[\![0,2,2]\!]$ and $[\![1]\!]$ by placing the complexes in the planar arc diagram corresponding to the product of two tangles, illustrated below. The tangles $\langle 0,2,2\rangle$ and $\langle 0,2,2,1\rangle$ are depicted in Figure 4.2.4.

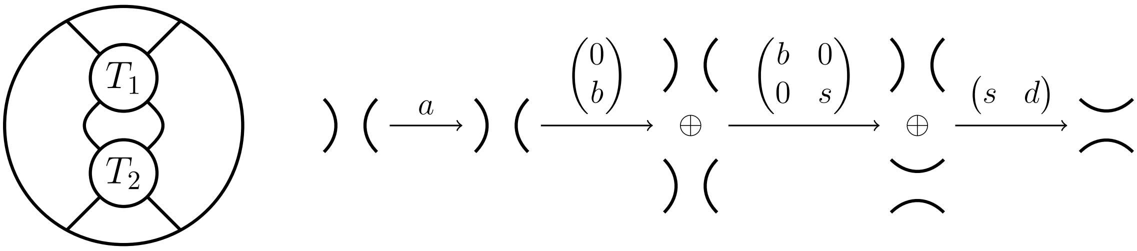
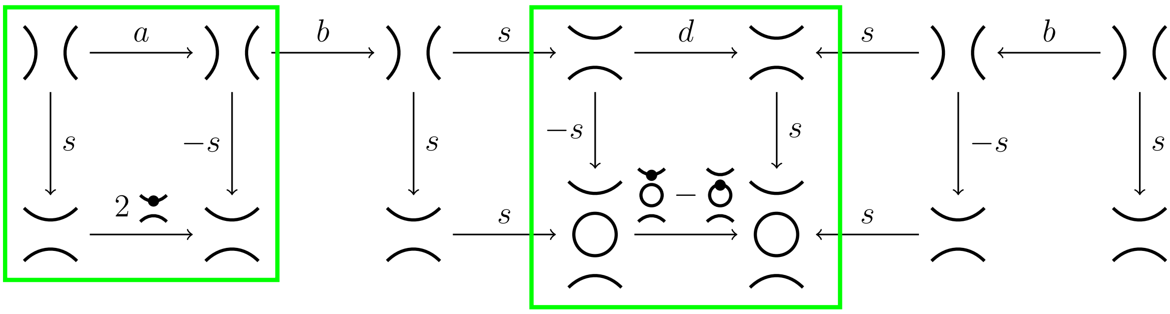


Figure 4.2.2: LEFT The ‘product’ planar arc diagram. RIGHT The complex $[\![0,2,2]\!]$ presented in the usual way. The dot diagram for the complex is in Figure 4.2.4 below.

Computing $D([\![0,2,2]\!],[\![1]\!])$ is annoying if we write $[\![0,2,2]\!]$ in the traditional way (as in Figure 4.2.2 above), since this introduces vectors and matrices into the complex. Obviously this poses no theoretical barrier, but it will be easier for illustrative purposes if we can depict the complexes in the simplest way possible. By treating the string presentation of $[\![0,2,2]\!]$ as we would a normal complex, we construct $D([\![0,2,2]\!],[\![1]\!])$ below.



One might at first be concerned by the presence of the backwards arrows and question whether we can still treat this as we would a normal complex. We can, since the diagram is simply an unwound form of the bulky complex that we would have otherwise obtained by using the traditional presentation of the complex.

As such, we can deloop and apply Gaussian elimination to our complex as we did in the previous section. (Though we need to be slightly more careful since subobjects may now have more than one arrow entering or leaving them.)

Both of the highlighted areas of the complex simplify. The left area simplifies after applying the square isomorphism, while the right area simplifies in the same way as when we worked with integer tangles in the last section. The result is Figure 4.2.3 below.

We began with a zig-zag complex that was the Khovanov complex of a rational tangle, and after adding a crossing to the tangle, the Khovanov complex simplified down until it was a zig-zag complex too. The overall change in the complex is illustrated in Figure 4.2.4.

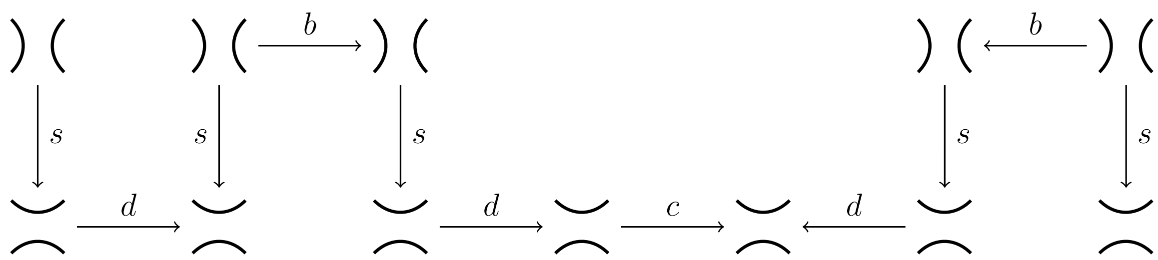


Figure 4.2.3: The simplified $[\![0, 2, 2, 1]\!]$ complex, after an application of the square isomorphism and two delooping / Gaussian elimination steps.

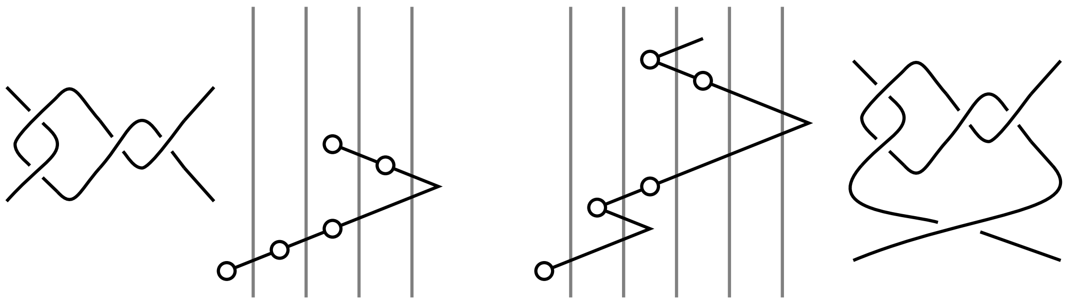


Figure 4.2.4: The tangles $\langle 0, 2, 2\rangle$, $\langle 0, 2, 2, 1\rangle$, and their Khovanov complexes. Notice that when a twist was added to the bottom of $\langle 0, 2, 2\rangle$, the number of $[0]$ tangles in the complex increased by the number of $[\infty]$ tangles.

We will see in the next subsection that the previous example is representative for rational tangles as a whole: that is, the Khovanov complex associated to a rational tangle is a zig-zag complex. Before we do this though, let us prove the square isomorphism is actually an isomorphism.

Proof of Proposition 4.2.1: The isomorphism and its inverse are in Figure 4.2.5 below. (The other terms in the complex are taken to each other via identity maps.) We need to check three things.

1. The bottom layer is actually a chain complex.
2. The collection of maps constitutes a chain map.
3. The chain maps are inverses of one another.

The first is easy – we just need to check that all the squares in the bottom layer anticommute. Only one of the squares is different from the top layer, so we need only check that $d \circ (\pm s) = -(\mp s) \circ 0 = 0$. This is true since the saddle in $d \circ \pm s$ connects both sheets in each of the cobordisms constituting d; the components then cancel.

The second point amounts to showing that the diagram in Figure 4.2.6 commutes.

(The other direction is similar.) This is easy and left as an exercise to the reader. (Hint: apply neck-cutting.)

The third point is easy as we merely need to check that

$$\begin{pmatrix} 1 & \mp s \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & \pm s \\ 0 & 1 \end{pmatrix}.$$

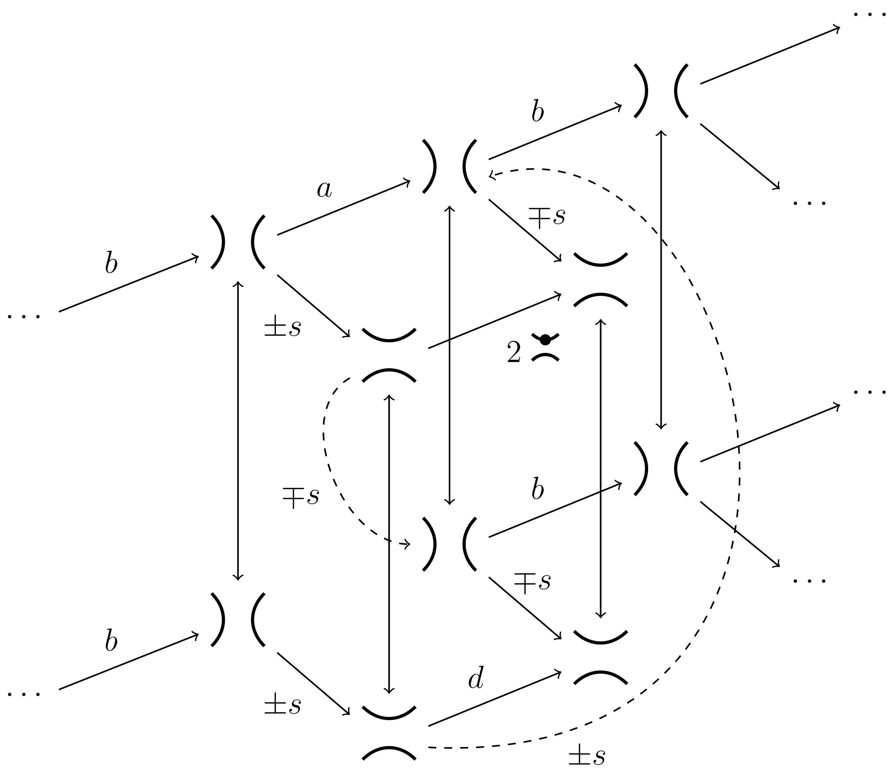


Figure 4.2.5: The square isomorphism, in all its glory. Unmarked vertical maps are identity maps. The identity of the other unmarked maps is irrelevant.

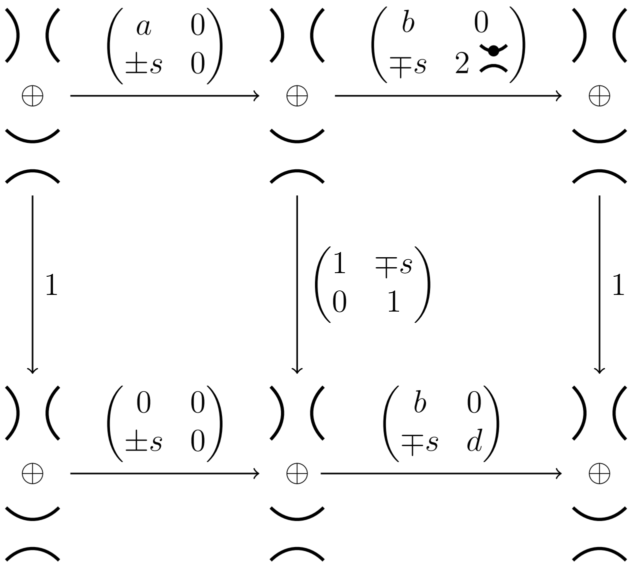


Figure 4.2.6: One direction of the square isomorphism, in a more traditional form.

4.3 The Khovanov complex of rational tangles

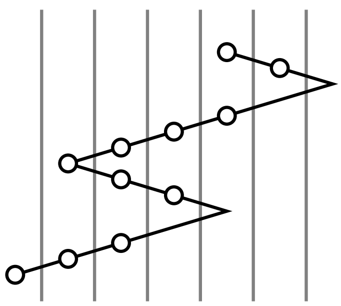
We claimed in the last subsection that $[\![T]\!]$ has the structure of a zig-zag complex for every rational tangle T. We now present a constructive proof. To avoid repeating ourselves, we will work with positive rational tangles ($F(T) > 0$); though all of what we say in this section can be analogously stated for negative rational tangles.

Recall from Section 2 that a positive rational tangle can be constructed from [1] by a finite sequence of tangle additions and products with [1]. When a rational tangle is positive, and positive crossings are added to T, the resulting Khovanov complex changes in a predictable way. This is essentially the content of Theorem 4.3.1. The proof follows by breaking up the complexes into several components, and applying the results earlier in the section to these components.

The description of how the complexes change that the theorem provides can almost be summarized by the pictures in Figure 4.3.1 below. If you can understand how the differences in the underlying tangles manifest in the pictures, then you've essentially understood the Khovanov homology of rational tangles.

For the remainder of the paper, we'll describe a zig-zag complex by its backbone of non-zero morphisms, writing these as a word. (Staring at one z-end of the zig-zag complex, we follow the backbone of non-zero morphisms until we reach the other z-end, writing down the morphisms along the way.) We call this word a *morphism string* of a zig-zag complex.

For example, the Khovanov complex given by



has morphism string

$$absdrb'a'babsdrb'.$$

Here a dash denotes a backwards arrow, and we have adopted the convention $r = s'$. In general, the morphism string of the Khovanov complex of a rational tangle is a word in the letters $\{a, a', b, b', c, c', d, d', s, r\}$.

Strictly speaking, the morphism string of a zig-zag complex isn't well defined – it depends on which z-end one starts at – but we'll adopt the convention that we begin at the z-end with the lowest homological height, if the two differ. (Our main result in this section doesn't depend on such a choice though – it holds for either morphism string.)

Morphism strings of the Khovanov complexes of rational tangles are not particularly readable, in that the structure of the complex is not as apparent as when it is represented as a dot diagram. As such we'll replace any instance of s with a negative-sloping diagonal line, and an r with a positive sloping diagonal line. With these conventions the morphism string above becomes

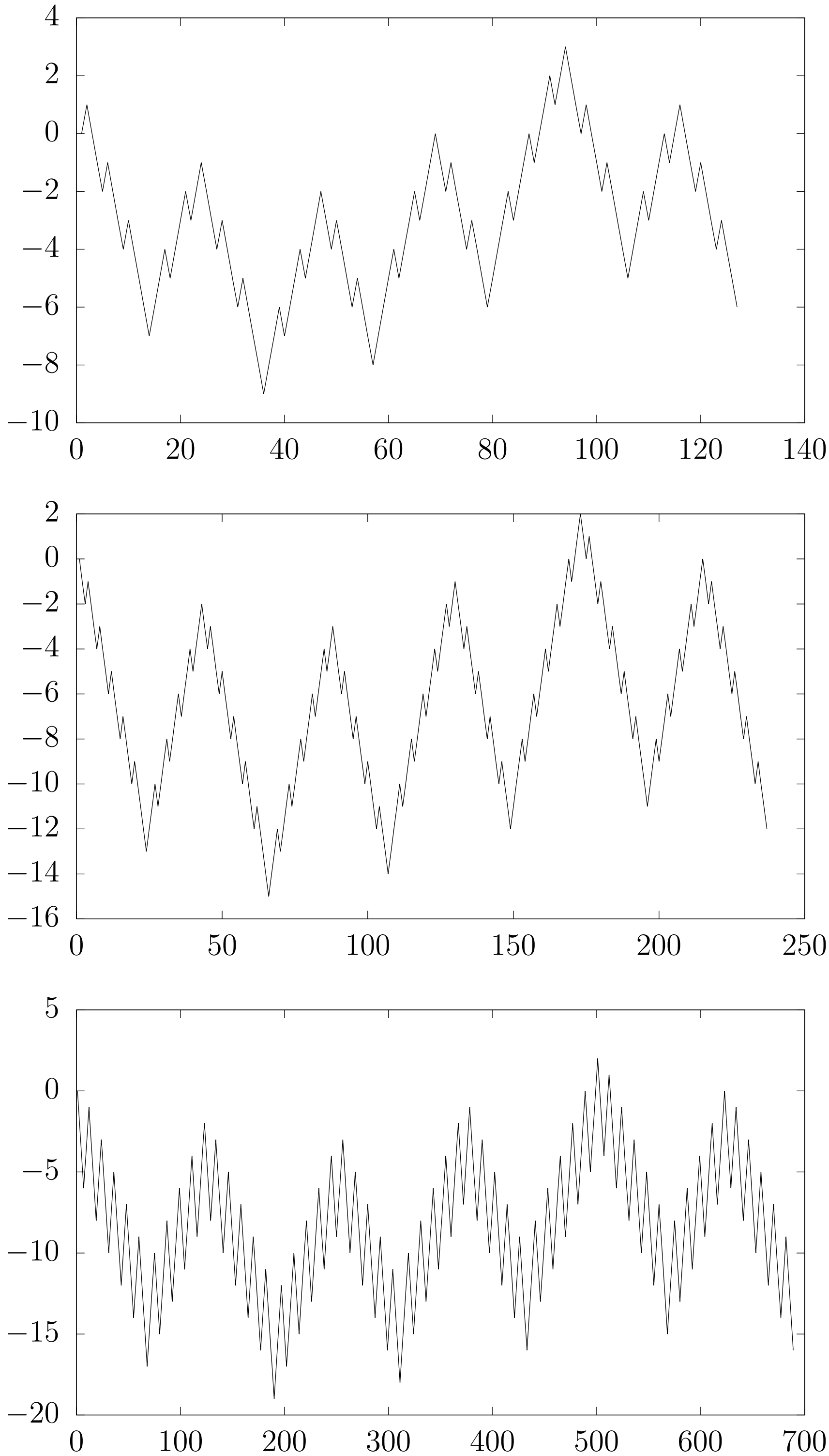


Figure 4.3.1: Top to bottom Dot diagrams for the complexes $[\![2,1,3,5,1]\!]$, $[\![2,1,3,10,1]\!]$, $[\![2,1,3,10,5]\!]$. Observe how the similarities and differences in the structure of the rational tangles manifest in the complexes. For readability we have removed the circles, displayed the diagrams so that the homological height is on the y-axis, and set the complexes to begin in homological height 0. The x-axis denotes the position of the subobjects in the string form of the complex.

$$\begin{tikzcd}[column sep=small]
ab \arrow[dr, no head] & & b'a'bab \arrow[dl, no head] \arrow[dr, no head] & & b'. \arrow[dl, no head] \\
& d & & d &
\end{tikzcd}$$

Writing the morphisms strings in this way is useful to illustrate how the complexes change when crossings are added to the underlying tangle, as well as a way to rapidly compute the Khovanov complex of a rational tangle by hand, as we do in Example 4.3.3.

Theorem 4.3.1 *Let T be a positive rational tangle. (The case of negative rational tangles is similar to what we present below.) Then the Khovanov complex $[\![T]\!]$ of T has a representative that is a zig-zag complex, and the morphism string associated to it is a word w in $\{a, b, c, d, r, s\}$, possibly with dashes on the letters $\{a, b, c, d\}$ satisfying the following condition. After removing the dashes from w, if $\tilde{w} = l_1 l_2 l_3$ is any subword of w consisting of three adjacent letters:*

- *if $l_2 = a$, then $\tilde{w} = bab$,*
- *if $l_2 = b$, then $l_1 \in \{a, r\}$, $l_3 \in \{a, s\}$,*
- *if $l_2 = c$, then $\tilde{w} = dcd$,*
- *if $l_2 = d$, then $l_0 \in \{c, s\}$, $l_3 \in \{c, r\}$,*
- *if $l_2 = r$, then $\tilde{w} = drb$,*
- *if $l_2 = s$, then $\tilde{w} = bsd$.*

Furthermore, the morphism string of $[\![T + [1]]\!]$ or $[\![T * [1]]\!]$ can be obtained from the morphism string of $[\![T]\!]$ by the following rules.

To obtain $[\![T + [1]]\!]$ from $[\![T]\!]$, split $[\![T]\!]$ into a list of subwords $w_1, \ldots, w_n$ (so that their concatenation $w_1 \cdots w_n$ is the morphism string) such that $w_i \in \{c,\ c',\ d,\ d',\ r\square,\ r\square s,\ \square s\}$ where $\square$ is a string in $\{a, a', b, b'\}$. The morphism string of $[\![T + [1]]\!]$ is given by the concatenation $f(w_1) \cdots f(w_n)$ where f is the following collection of rules.

- *If $\boxdot$ is the string obtained from $\square$ by replacing each letter $a/a'/b/b'$ with $b/b'/a/a'$ respectively, then*
  - $f(r\square) = rb'\boxdot$,
  - $f(r\square s) = rb' \boxdot bs$,
  - $f(\square s) = \boxdot bs$.
- *If $w_i = d,\ d'$,*
  - $f(w_i) = sw_i \quad (i = 1)$,
  - $f(w_i) = w_i \quad (1 < i < n)$,
  - $f(w_i) = w_i r \quad (i = n)$.
- *If $w_i = c, c'$, $f(c) = rbs$, $f(c') = rb's$.*

To obtain $[\![T * [1]]\!]$ from $[\![T]\!]$, split $[\![T]\!]$ into a list of subwords $w_1, \ldots, w_n$ (so that their concatenation $w_1 \cdots w_n$ is the morphism string) such that $w_i \in \{a,\ a',\ b,\ b',\ s\square,\ s\square r,\ \square r\}$ where $\square$ is a string in $\{c, c', d, d'\}$. The morphism string of $[\![T * [1]]\!]$ is given by the concatenation $g(w_1) \cdots g(w_n)$ where g is the following collection of rules.

- *If $\boxdot$ is the string obtained from $\square$ by replacing each letter $c/c'/d/d'$ with $d/d'/c/c'$ respectively, then*
  - $g(s\square) = sd\boxdot$,

 - $g(s\square r) = sd \boxdot d'r$,
 - $g(\square r) = \boxdot d'r$.

- *If* $w_i = b,\ b'$,
  - $g(w_i) = rw_i \quad (i = 1)$,
  - $g(w_i) = w_i \quad (1 < i < n)$,
  - $g(w_i) = w_i s \quad (i = n)$.
- *If* $w_i = a, a'$, $g(a) = sdr$, $g(a') = sd'r$.

Before embarking on the proof, we look at a few examples.

Example 4.3.2 Let us check that these rules agree with the calculation in Example 4.2.3. Examining Figure 4.2.4, we see that the morphism strings of $[\![0,2,2]\!]$ and $[\![0,2,2,1]\!]$ are $absdrb'$ and $sdrbsdcd'rb's$ respectively.

Since $\langle 0,2,2,1\rangle$ is equal to $\langle 0,2,2\rangle * [1]$, we can compute the morphism string of $[\![0,2,2,1]\!]$ from the morphism string of $[\![0,2,2]\!]$. Following the description in Theorem 4.3.1, we break up $[\![0,2,2]\!]$ into subwords:

$$absdrb' = a + b + sdr + b'.$$

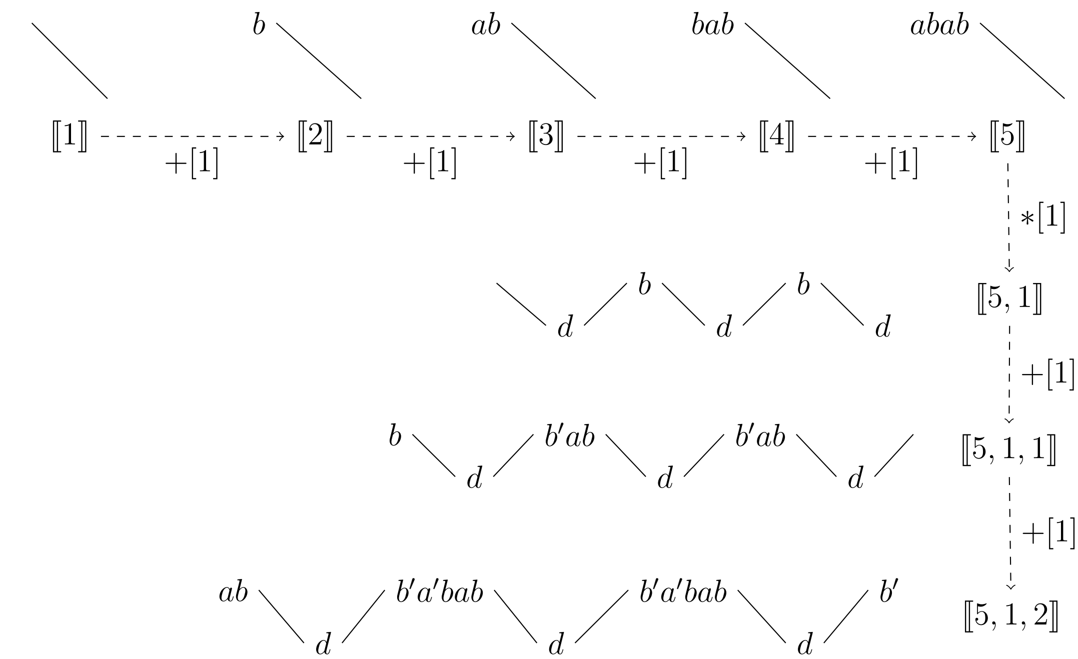
Applying the rules we have $g(a) = sdr$, $g(b) = b$, $g(sdr) = sdcd'r$ and $g(b') = b's$. Concatenating these, we have

$$g(a)g(b)g(sdr)g(b') = sdrbsdcd'rb's.$$

Example 4.3.3 Let us compute $Kh(8_2)$ by hand. This will further illustrate how the Khovanov bracket of a rational tangle changes when additional crossings are added to the underlying tangle.

The knot 8_2 is rational, obtained via numerator closure (Section 2.3) from $\langle 5,1,2\rangle$. Let us compute the structure of the chain complex first; gradings and homological heights will be calculated at the end.

The morphism string corresponding to $[\![1]\!]$ is just s. By applying the rules above and using the notation mentioned earlier, we build the morphism string of $[\![5,1,2]\!]$ from s as follows.



All that's left to do is to take the numerator closure of this complex and determine the homological and grading information. The former is easier to do if we construct the dot diagram of the complex. This and its closure are drawn below in Figure 4.3.2.

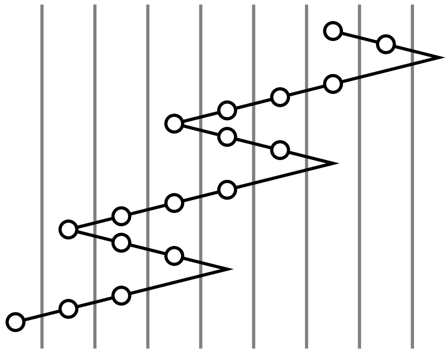
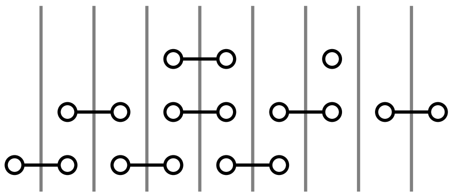


Figure 4.3.2: The dot diagram of $[\![5,1,2]\!]$ and $N([\![5,1,2]\!])$. This example is illustrative of the Khovanov homology of rational knots in general.

By a similar argument as that in Example 4.1.3, the wedge-shaped complexes in the dot diagram with four subobjects are homotopy equivalent to  components.

Therefore $[\![8_2]\!]$ splits into a direct sum of nine complexes which span the homological heights as illustrated below. We have already computed the homology of these complexes in Example 4.1.3, so are nearly finished. We simply need to calibrate the homological heights and gradings.

It is a simple exercise to verify that the internal grading and homological height of one of the subobjects of the complex determines the same grading information for the rest of the subobjects. As such we only need to compute this grading information for one subobject in the complex to have determined the grading information of $[\![5,1,2]\!]$.



In the next section, we will see that this bigrading information can simply be determined by multiplying certain 2×2 matrices together (Theorem 5.0.4), but for now we will use the calculation so far.

An orientation of 8_2 orients the underlying tangle. To calibrate both the homological and quantum grading information of $[\![5,1,2]\!]$, we keep track of the grading information associated to the z-end at the start of the morphism string of each intermediate complex in the construction of $[\![5,1,2]\!]$. Determining how the grading information of this subobject changes when crossings are added to the underlying tangle can be done by examining: the morphism leaving the subobject; the orientation of the crossing being added; whether the crossing is being added to the to the right or bottom of the tangle. The relative change in the grading information for positively oriented crossings is listed in the table below.

The corresponding values for negatively-oriented crossings differ from their positively-oriented counterparts by $-3/-1$. After applying this process to our construction of the morphism string of $[\![5,1,2]\!]$ above, we find that the first subobject in string of $[\![5,1,2]\!]$ has grading/homological height $-16/-6$.

Map	$+[1]$	$*[1]$
a, a'	$0/0$	$1/0$
b, b'	$0/0$	$2/1$
c, c'	$2/1$	$-1/0$
d, d'	$1/0$	$-2\ /\ 1$
s	$0/0$	$-3/-1$
r	$2/1$	$-1/0$

Table 1: The relative shifts in quantum grading (i) and homological height (h) of the subobject is presented as i/h. These values are for positively oriented crossings.

For all intents and purposes we're done, though it's common to write the result as the Khovanov polynomial instead of the actual Khovanov homology. One is more succinct than the other and they essentially contain the same information.

The Khovanov polynomial of a link is a two-variable Laurent polynomial in $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ defined by

$$\sum_j \mathrm{qdim}(Kh^j(L)) \cdot t^j. \tag{4.3.1}$$

It is common to denote the Khovanov polynomial in the form of a table. The positive integer in i-th j-th square of the table corresponds to the coefficient of $t^j q^i$ in the polynomial.

Filling the table in using the information obtained above is a trivial matter; ○—○ components in the dot diagram correspond to knights moves while the ○ corresponds to the exceptional pair. (This correspondence follows from the calculations in Example 4.1.3.)

$i \setminus j$	-6	-5	-4	-3	-2	-1	0	1	2
1									1
-1									
-3							2	1	
-5						1	1		
-7					2	1			
-9				1	1				
-11			1	2					
-13		1	1						
-15		1							
-17	1								

Table 2: The Khovanov homology of $Kh(8_2)$.

We now return to the main result of this section.

Proof of Theorem 4.3.1: The first claim of the theorem regarding the structure of $[\![T]\!]$ for rational T follows from induction from the rules of the theorem. Namely, since every rational tangle is given by a finite sequence of additions and products of [1], one simply shows that the rules preserve the property of the first claim.

The proof of the rules essentially follows from thinking about how the complexes associated to the morphism strings of the subwords in the rules change when the underlying

tangle changes. Proposition 4.1.1 and the isomorphism in Proposition 4.2.1 are sufficient to describe these changes.

As a sanity-check we wrote a computer program in Python that calculates $[\![T]\!]$ for positive rational tangles T based on our rules. We then extended the program to calculate $Kh(R)$ for rational R. We checked the results for the first 25 rational knots; they are all correct.[2]

The proof of the rules describing how the morphism string of $[\![T + [1]]\!]$ is obtained from $[\![T]\!]$ is analogous to the proof of the rules for the $[\![T * [1]]\!]$ case; as such we prove the latter. If the reader is not already familiar with Example 4.2.3 we suggest she look at it now; it is easy to use this as a visual guide to explain the rules.

Let us first explain the general structure of $[\![T * [1]]\!]$. If the morphism string of $[\![T]\!]$ contains n letters, can view $[\![T*[1]]\!]$ as a '$2\times(n+1)$ complex' just as we did in the previous section. That is, by constructing the complex using the product planar arc diagram, we have $[\![T * [1]]\!] = [\![D([T], [1])]\!] = D([\![T]\!], [\![1]\!])$. This is a complex consisting of two rows, each with $n + 1$ subobjects, and various left, right, and down arrows between adjacent subobjects in the complex. The arrows create squares, and each of these anticommute. We now simplify the complex.

By dropping arrows that are zero morphisms, one can view the complex $[\![T * [1]]\!]$ as a series of chains of anticommuting squares 'connected' by b maps. (For instance, in the aforementioned example, the complex consists of two such chains of squares; one chain consists of one square, the other consists of three squares.) Each of these chains of squares will simplify so that each subobject they contain will have precisely two non-zero maps coming in or out of it.

By virtue of the morphism string structure of $[\![T]\!]$, there are two types of these chains of squares: single-square chains containing an a or a' map, and chains which contain tangles with loops. The first type of square chains simplify by the square isomorphism of Proposition 4.2.1, the other type simplify via the proof of Proposition 4.1.1.

One then sees that the square chains containing an a or a' simplify to have morphism string sdr or $sd'r$ respectively, the third rule regarding $[\![T * [1]]\!]$. The other type of square chains simplify to give the first rules of $[\![T * [1]]\!]$. (Note that the first and last rule of this group of three rules describe the cancellation if the square forms the beginning or end of the $2 \times (n + 1)$ complex respectively.)

The only rule left to describe is that regarding the b maps; since these 'connect' the square of chains that simplify, they do not change, save if they are at the end or beginning of the complex. □

5 Bigradings and the reduced Burau representation of B_3

The main result of the previous section, Theorem 4.3.1, showed how the homotopy class of the Khovanov complex of a rational tangle has a particularly simple representative that is a zig-zag complex (Definition 1.0.1). The theorem described how the structure of the complex changed when crossings are added to the underlying tangle in terms of its morphism string. But this is only half the picture: we did not describe how the bigradings of the subobjects in the complex change. (That is, how the homological heights and quantum gradings of the subobjects change.)

[2] The code is available at `https://tqft.net/research/BenjaminThompson`

Quite surprisingly, it turns out that these changes can be described by matrix actions, (Theorem 5.0.4), and furthermore, this action, after a change of basis, is the reduced Burau representation of B_3 (Proposition 5.0.6).

We first formalize what we mean by 'bigrading information'. Let BN(4) be the graded subcategory of $\mathrm{Mat}(\mathcal{C}ob^3_{\bullet/l})$ (Section 3.2) generated by the [0] and [∞] tangles. For any rational T, an arbitrary representative in the homotopy class $[\![T]\!]$ may not reside in Kom(BN(4)), but the canonical representative of $[\![T]\!]$ as described by Theorem 4.3.1 is in Kom(BN(4)).

For the purposes of the next definition, we will assume that BN(4) is Krull-Schmidt. (That is, every object is isomorphic to a finite direct sum of indecomposable objects, and this decomposition is unique up to isomorphism.) The validity of this claim doesn't affect our results, though by assuming it the descriptions below are cleaner.

Define a function $\Psi : \mathrm{Obj}(\mathrm{Kom}(BN(4))) \to \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]\langle \asymp,)(\rangle$ as follows. Given a complex Ω, express each of its objects as direct sum of indecomposable objects. To each indecomposable object X in Ω^j we associate the element $q^i t^j X$ where i is the internal grading of the subobject. Then define $\Psi(C)$ to be the sum of these elements over all subobjects in the complex.

We will present an element $p_0 \asymp + p_\infty)(\in \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]\langle \asymp,)(\rangle$ as (p_0, p_∞). For example,

$$\Psi(\bullet \to \underline{)(}_{(1)} \longrightarrow \asymp_{(2)} \longrightarrow \asymp_{(4)} \to \bullet) = (q^2 t + q^4 t^2, q).$$

Note that Ψ is not a homotopy invariant. (For example, $\Psi(\bullet \to \underline{)(}_{(0)} \xrightarrow{\phi})(_{(0)} \to \bullet) = (0, 1+t)$, even though this complex is contractible when ϕ is an isomorphism.) As such, to make the map well-defined we will only consider complexes in Kom(BN(4)) up to isomorphism.

Let us now consider rational tangles as partial closures of elements in B_3. (See Figure 2.1.6.) More precisely, if a positive (negative) rational tangle T has standard form $\langle a_1, a_2, \ldots, a_n \rangle$ with $a_i \geq 0$ ($a_i \leq 0$), we can consider it as the element $\sigma_1^{a_1}\sigma_2^{-a_2}\sigma_1^{a_3}\sigma_2^{-a_4}\cdots$ of B_3 whose bottom two rows have been closed off at the beginning.

We note that this isn't the traditional way to close off a braid. Rather, our 'closure' of a braid is really the numerator closure of the corresponding rational tangle. (For instance, the figure-8 knot is the (numerator) closure of $\sigma_1\sigma_2^{-1}\sigma_1^2$, as illustrated in Figure 2.3.1.)

From such a perspective, adding [1] to a rational tangle corresponds to multiplying the braid by σ_1, while multiplying the tangle by [1] corresponds to multiplying the braid by σ_2^{-1}.

We know from Corollary 2.2.9 that every rational tangle has a unique canonical form; from this it follows that each positive (negative) rational tangle has a unique presentation $\sigma_1^{a_1}\sigma_2^{-a_2}\sigma_1^{a_3}\cdots\sigma_1^{a_n}$ with $a_i > 0$ ($a_i < 0$) for $i < n$ and $a_n \geq 0$ ($a_n \leq 0$).

We can then describe the bigrading information of the Khovanov complexes of positive rational tangles as follows. (Negative rational tangles admit a similar description.)

Theorem 5.0.4 *Let $\phi : \mathbb{Q} \to B_3$ be the function taking a rational number r to the canonical braid (described above) corresponding to the rational tangle T with $F(T) = r$. Let Ψ be as above. For any rational tangle T, identify $[\![T]\!]$ with its canonical representative in* Kom(BN(4)) *as described by Theorem 4.3.1. Finally, let $r \in \mathbb{Q}_+$, and fix an orientation of $\phi(r)$. Then $\Psi([\![\phi(r)\sigma_1]\!])$ is given by $\Psi([\![\phi(r)]\!])R_\pm$ and $\Psi([\![\phi(r)\sigma_2^{-1}]\!])$ is given by $\Psi([\![\phi(r)]\!])B_\pm$, where*

$$R_+ = q\begin{pmatrix} qt & 1 \\ 0 & q^{-1} \end{pmatrix}, \qquad R_- = q^{-2}t^{-1}\begin{pmatrix} qt & 1 \\ 0 & q^{-1} \end{pmatrix},$$

$$B_+ = q^2 \begin{pmatrix} qt & 0 \\ t & q^{-1} \end{pmatrix}, \qquad B_- = q^{-1}t^{-1} \begin{pmatrix} qt & 0 \\ t & q^{-1} \end{pmatrix}.$$

(The signs correspond to the orientation of the crossing being added.)

We note that this is quite unexpected. While B_3 acts on rational tangles as described, there is no reason to expect that the bigrading information of the minimal (reduced) Khovanov complexes should admit such a description.

Example 5.0.5 Let us check our example agrees with the calculation of $Kh(8_2)$ in Example 4.3.3. The knot 8_2 is the numerator closure of $\langle 5,1,2\rangle$. When 8_2 is oriented the underlying tangle inherits an orientation of type I (see the end of Section 2.3). As such, when we build $\langle 5,1,2\rangle$ from $[0]$, the first six crossings have negative orientation, while the last two crossings are positively oriented.

As such, the bigrading information of $[\![5,1,2]\!]$ is given by

$$(1,0)\cdot {R_-}^5 B_- {R_+}^2 = (p_0, p_\infty).$$

Calculating these, we find

$$p_0 = q^{-11}t^{-3} + q^{-9}t^{-2} + q^{-7}t^{-1} + q^{-5} + q^{-3}t + q^{-1}t^2,$$
$$p_\infty = q^{-16}t^{-6} + 2q^{-12}t^{-5} + 3q^{-12}t^{-4} + 3q^{-10}t^{-3} + 3q^{-8}t^{-2} + 2q^{-6}t^{-1} + 2q^{-4} + q^{-2}t.$$

As expected from the previous calculation, we find that the subobject with the lowest bigrading has quantum grading -16 and homological height -6. By examining Figure 4.3.2 we see that bigrading information of the other subobjects in the complex are accounted for.

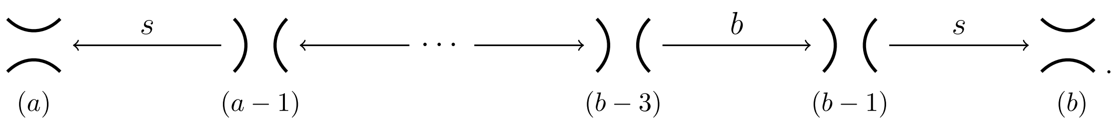
Before saying anything further, let us prove the result.

Proof of Theorem 5.0.4: We will prove that $\Psi([\![\phi(r)\sigma_1]\!])$ is given by $\Psi([\![\phi(r)]\!])R_\pm$. (The proof of the other claim is analogous.) Let T be a positive rational tangle, and consider $[\![T]\!]$. Adding $[1]$ to T is the same as multiplying the corresponding braid by σ_1. We will prove the claim by examining how the bigradings of the subobjects in $[\![T]\!]$ influence the bigradings of the subobjects in $[\![T+[1]]\!]$. By doing this we will be able to express the bigrading information of $[\![T+[1]]\!]$ in terms of $[\![T]\!]$.

From Theorem 4.3.1 we can uniquely split the morphism string of $[\![T]\!]$ into subwords in $\{c, c', d, d', r\square, r\square s, \square s\}$. We can account for all the subobjects in $[\![T+[1]]\!]$ by accounting for the subobjects created by each subword of $[\![T]\!]$, though when we do this we need to be careful not to over-count or miss any subobjects.

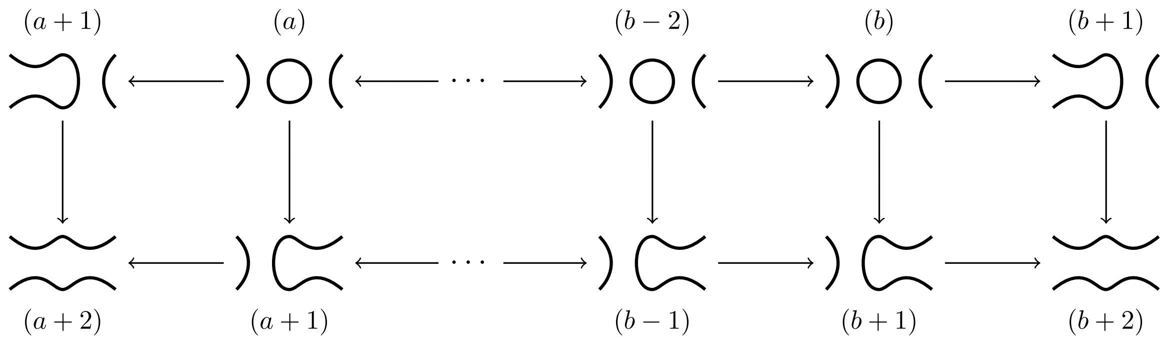
As in the last section, we construct $[\![T+[1]]\!]$ from $[\![T]\!]$ using the 'addition' planar arc diagram D. That is, $[\![T+[1]]\!] = [\![D(T,[1])]\!] = D([\![T]\!],[\![1]\!])$. Let us further assume that the orientation of the crossing being added is positive. (So we will need to show the bigradings change in a manner described by the action of R_+.)

Theorem 4.3.1 tells us that a word in the morphism string of $[\![T]\!]$ of the form $r\square s$ is transformed to $rb' \boxdot bs$. (In this instance $\square$ is an alternating word in $\{a, a', b, b'\}$.) Let us recall the proof. The complex associated to $r\square s$ has the form

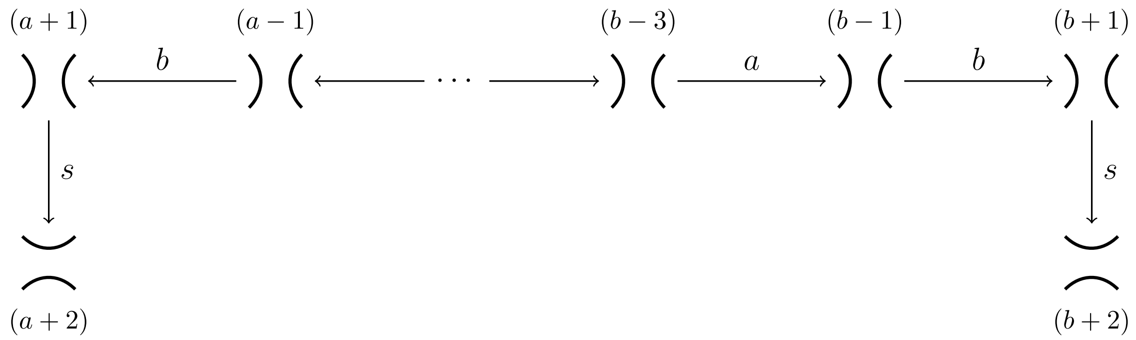


This consists of n)(subobjects and 2 $\asymp$ subobjects.

Recall that when [1] is positively oriented, $[\![1]\!] = \bullet \to \underline{)(}_{(1)} \longrightarrow \asymp_{(2)} \to \bullet$. Hence when $D([\![T]\!], [\![1]\!])$ is constructed, the previous complex segment gives rise to the following complex segment.



After multiple applications of delooping and Gaussian elimination, this simplifies down.



There are now $n+2$)(subobjects, and still 2 $\asymp$ subobjects. However, the $\asymp$ subobjects now differ from the $\asymp$ subobjects in the first complex by a factor of q^2t.

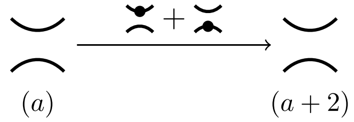
On the other hand, of the $n+2$)(subobjects of the top row above, n were produced after delooping $D()(,)()$ terms. These subobjects picked up a factor of q from $[\![1]\!]$, but then immediately lost it after delooping. The two other)(subobjects were produced via $D()(, \asymp)$. The bigrading information of these will then differ from those of the $\asymp$ subobjects in the first complex by a factor of q.

It follows, then, that the bigrading information (p'_0, p'_∞) of this segment can be described in terms of the bigrading information of the first segment (p_0, p_∞) via

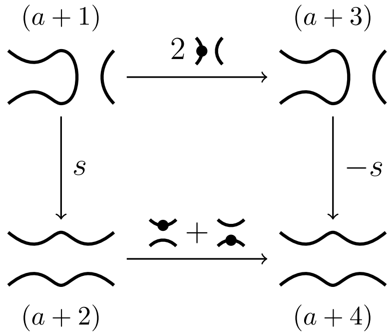
$$(p'_0, p'_\infty) = (p_0, p_\infty) \cdot \begin{pmatrix} q^2t & q \\ 0 & 1 \end{pmatrix} = (p_0, p_\infty) \cdot R_+.$$

By the same argument the change in bigrading information described by $r\square \to rb'\boxdot$ and $\square s \to \boxdot bs$ is captured by multiplication by R_+.

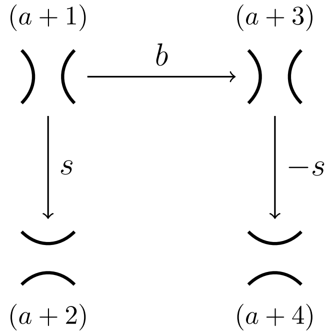
Now the words between words of the form $r\square s$ in the morphism string for $[\![T]\!]$ are in $\{d, d, c, c'\}$. So consider the subcomplex associated to c.



When $D([\![T]\!], [\![1]\!])$ is constructed, this gives rise to the following complex segment.



This simplifies by the square isomorphism.



The bigrading of the subobjects here differ from the originals by a factor of q^2t, while the bigrading of the subobjects differ from the originals by a factor of q. Hence, on this complex segment the overall change in bigrading information can be described by the action of R_+.

We're essentially done: if a subword d/d' in the morphism string is not at either end, then it is adjacent to letters in $\{r, s, c, c'\}$, meaning the subobjects and the head and tail of d have already been accounted for by our analysis of the $r\square s \to rb' \boxdot bs$ and $c \to rbs$ transformations. If d/d' is at the end of the morphism string, then there will be a subobject and the head or tail of it we will not have accounted for yet, but it is easy to check that the bigradings of the two subobjects in $[\![T + [1]]\!]$ that this produces can be described by the action of R_+ on the bigrading information of the subobject too.

Hence the change in bigrading information of the complex is locally described by the action of R_+. Since we have accounted for all the subobjects in $[\![T + [1]]\!]$, it follows that $\Psi([\![\phi(r)\sigma_1]\!])$ is given by $\Psi([\![\phi(r)]\!])R_+$ when the crossing being added is has positive orientation. We then immediately get that $\Psi([\![\phi(r)\sigma_1]\!])$ is given by $\Psi([\![\phi(r)]\!])R_-$ when the crossing being added has negative orientation, since $\Psi([\![1]\!])$ when $[1]$ has negative orientation differs from $\Psi([\![1]\!])$ when $[1]$ has positive orientation by a factor of $q^{-3}t^{-1}$. $\square$

Given that the bigrading information admits such an elegant description in terms of matrix actions and B_3, a natural question to ask is whether the matrices satisfy the braid relation. Since multiplying the braid by σ_1 corresponds to multiplication by $R_\pm$, and multiplying the braid by σ_2^{-1} corresponds to multiplication by $B_\pm$, we would expect $R_\pm$ and $B_\pm^{-1}$ to braid. This is indeed the case.

Proposition 5.0.6 *The pairs of matrices (R_+, B_-^{-1}) and (R_-, B_+^{-1}) satisfy the braid relation. After a change of basis these give the (reduced) Burau representation of B_3.*

Proof: With
$$p = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix},$$
when we change basis we obtain
$$\sigma_1' = p^{-1}R_+p = \begin{pmatrix} q^2t & 1 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_2' = p^{-1}B_-^{-1}p = \begin{pmatrix} 1 & 0 \\ -q^2t & q^2t \end{pmatrix}.$$
This is exactly the (reduced) Burau representation of B_3 after the change of variables $t \mapsto -q^2t$. $\square$

We note however, that while $R_\pm$ and $B_\pm$ describe the addition or product of [1] with a positive rational tangle, $R_\pm^{-1}$ and $B_\pm^{-1}$ *do not* describe the addition or product of $[-1]$ with a positive rational tangle.

For instance, consider the Khovanov complex associated to [1] with positive orientation. When the corresponding braid (σ_1) is multiplied by σ_1^{-1}, the [0] tangle is obtained. However,
$$\Psi(\llbracket 1 \rrbracket) \cdot R_-^{-1} = (q^2t, q) \cdot \begin{pmatrix} q & -q^2 \\ 0 & q^3t \end{pmatrix} = (q^3t, 0) \neq \Psi(\llbracket 0 \rrbracket) = (1, 0).$$
We note though, that the orientations are at fault for the discrepancy here: if [1] is positively oriented, then when the braid is multiplied by σ_1^{-1}, the new crossing is negatively oriented. (Indeed, R_+ and R_- differ from each other by a factor of q^3t.)

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